THE INFINITE FERN IN HIGHER DIMENSIONS

by

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1. Introduction

A general question in the Langlands program is the relation between automorphic representations and Galois representations. In such a generality the question is completely open, but we can restrict to an apparently simpler question : can we relate deformations on both sides ? In fact, there is a natural geometric interpretation of this question as follows. Assume that E is a number field, and

$$\overline{\rho} : \mathrm{Gal}(\overline{E}/E) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{F}}_p),$$

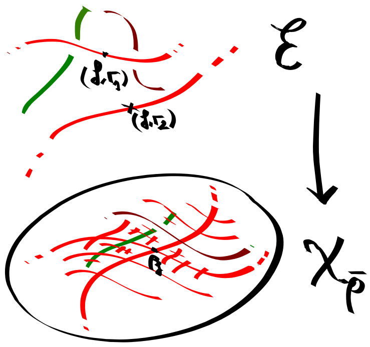
is a continuous (i.e. which factors through a finite extension) representation. The deformations of $\overline{\rho}$, with minor technical conditions, with values in finite extensions of $\mathbb{Q}_p$, can be arranged in a natural rigid space $\mathcal{X}_{\overline{\rho}}$, by work of Mazur. Conjecturally, and now in many known cases ([**CHLN11**, **CH13**, **Shi14**, **HLTT16**, **Sch15**]) automorphic representations give rise to Galois representations, some of which gives points in $\mathcal{X}_{\overline{\rho}}$, that we call ***of automorphic nature***, or just ***automorphic***. It is then natural to wonder which structure have these automorphic points in $\mathcal{X}_{\overline{\rho}}$: is it an algebraic subspace ? a closed one ? is it Zariski dense ?

The first example after the case of characters was studied by Gouvêa-Mazur. In this situation $E = \mathbb{Q}$ and $n = 2$, and $\overline{\rho}$ is irreducible, modular and unobstructed, so that $\mathcal{X}_{\overline{\rho}}$ is a 3-dimensional open ball. In this situation automorphic points are related to modular forms. In [**GM98**], Gouvêa and Mazur show that automorphic points are Zariski dense in $X_{\overline{\rho}}$ using the so called *infinite fern*. Let us explain this name : up to twisting by powers of the cyclotomic character, we can replace $\mathcal{X}_{\overline{\rho}}$ by a two dimensional open ball. Modular forms (of finite slope) can be interpolated by a geometric object, the Coleman-Mazur *Eigencurve* $\mathcal{E}$ ([**CM98**]), which is a rigid-analytic curve, whose points are refined p-adic modular forms of finite slope. Generically, a classical modular forms has two refinements, thus gives rise to two distinct points in the Eigencurve. Moreover the points corresponding to refined classical modular form are Zariski dense in the Eigencurve. By p-adic interpolation, it is possible to associate a 2-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to a point of $\mathcal{E}$.

The points giving rise to deformations of $\overline{\rho}$ form a union $\mathcal{E}_{\overline{\rho}}$ of connected components of $\mathcal{E}$ and the universal property of $\mathcal{X}_{\overline{\rho}}$ implies the existence of a map

$$\mathcal{E} \longrightarrow \mathcal{X}_{\overline{\rho}}.$$

Generically, each modular point f in $\mathcal{X}_{\overline{\rho}}$ has two preimage in $\mathcal{E}$, giving rise to two distincts small curves around those preimages, whose image in $\mathcal{X}_{\overline{\rho}}$ meet only at f. By density, each of these two small curves has a Zariski dense set of modular points, and for each of these points there is another small curve passing through, and so on, giving a fractal-like object which we picture as follows :



giving a justification for the name of the *infinite fern*.

This article deals with a generalization of this result to more general number fields and greater values of n. First, we need to assume that the number field E is a CM field, with totally real field F, in order to be able to associate Galois representations to automorphic representations. Second, it is expected that for general n the automorphic points are not Zariski dense in $\mathcal{X}_{\overline{\rho}}$, thus we reduce to the case of χ-*polarized* Galois representations, for a character $\chi : G_E \longrightarrow \overline{\mathbb{Q}_p}^{\times}$, i.e. continuous group homomorphisms $\rho : \mathrm{Gal}(\overline{E}/E) \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that

$$\rho^\vee \simeq \rho^c \otimes \chi\varepsilon^{n-1}, \quad \text{where } \rho^c := \rho(c \cdot c^{-1}),$$

where ε is the cyclotomic character, and $c \in \mathrm{Gal}(\overline{E}/F)$ is a lift of the unique non trivial element of $\mathrm{Gal}(E/F)$. Fix S a finite set of primes of E containing all primes above p. In this situation, assume that $\overline{\rho}$ is χ-polarized, absolutely irreducible (for simplicity) and unramified away from S. Let be the complete noetherian local algebra $R_{\overline{\rho}}^{\chi-pol}$ parametrizing deformations of $\overline{\rho}$ which are χ-polarized and unramified away from S. Its rigid fiber $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ is a rigid space of dimension at least $[F : \mathbb{Q}]\frac{n(n+1)}{2}$. A natural source of automorphic points in $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ is given by the regular, algebraic, essentially polarized, cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E)$, by work of many authors ([**HT01**, **CHT08**, **CHLN11**, **CH13**, **Shi14**] for example). In this paper, we make the following hypothesis,

Hypothesis 1.1. — 1. $\overline{\rho}$ is conveniently modular (see Definition 4.1),
2. All primes above p in F are unramified, and split in E,

3. χ is crystalline at p, satisfies $\chi = \chi^c$ and satisfies a sign condition (see Hypothesis 3.7 and section 2)

Under the previous hypothesis, we have the following result,

Theorem 1.2. — *The Zariski closure of automorphic points contains a (non empty) union of irreducible components of $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$, each of which are of dimension $[F:\mathbb{Q}]\frac{n(n+1)}{2}$.*

We say that our deformation problem is unobstructed if $H^2(G_{F,S},\mathrm{ad}(\overline{r})) = \{0\}$ where $\overline{r}$ is some extension of $\overline{\rho}$ to $G_{F,S}$ the Galois group of the maximal unramified extension of F (see section 3). In this situation, we know that $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ is a rigid open unit ball in $[F:\mathbb{Q}]\frac{n(n+1)}{2}$-variables.

Corollary 1.3. — *Under the previous hypothesis, if moreover $\overline{\rho}$ is unobstructed, then automorphic points are Zariski dense in $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$.*

Remark 1.4. — In [**Gui20**] Giraud proved that if π is an extremely regular automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ (see [**BLGGT14**] section 2.1), then there exists a density 1 set of primes λ of E such that $\rho_{\pi,\lambda}$ is unobstructed. As we have assumed $\overline{\rho}$ to be conveniently modular, we can actually find some extremely regular π, so that $\overline{\rho} = \overline{\rho}_{\pi,\lambda}$ for $\lambda|p$, and thus using [**Gui20**], up to change λ in a density 1 set, we can assume unobstructedness. In particular, under our assumption, $\overline{\rho}$ is part of a compatible system for which in a density 1 set of primes, we have Zariski density of automorphic points in the associated deformation spaces.

Before explaining the strategy of proof, let us say what was known. The first case was the case non-polarised, $n = 2$ and $E = F = \mathbb{Q}$, when unobstructed, which was proven by Gouvêa-Mazur [**GM98**], and generalised by Boeckle [**BÖ01**]. The non-polarised case for $n = 2$ and totally real fields $E = F$, and the polarised case of $n = 3$ (and general CM fields E/F) was proved by Chenevier ([**Che11**]). A generalisation for greater n; but under more restrictive hypothesis (of Taylor-Wiles type) on E and $\overline{\rho}$, was proven recently by Hellman-Margerin-Schraen ([**HMS22**]). All of these proofs uses the analogue (in higher dimensions) of the infinite fern. We know try to explain our strategy together with the relation to the previous works.

The Galois representations that we study can be viewed as p-adic ($L-$ or $C-$) parameters of reductive group: $\mathrm{GL}_2/\mathbb{Q}$ in the situation of Gouvêa-Mazur and $U_{E/F}(n)$, or $GU_{E/F}(n)$, or one of its inner forms, a (similitude) unitary group in n-variables, for polarized deformation problems. A natural source of automorphic points is given by automorphic representations of these groups. It turns out that these groups give rise to Shimura data, that we can use to construct p-adic refined families of automorphic forms, that is p-adic automophic eigenforms together with the extra data of a refinement. These families generalize the Eigencurve of Gouvêa-Mazur and are called Eigenvarieties (see [**CM98**, **Che04**, **Urb11**, **Eme06**, **AIP15**, **Her19**] for example). It turns out that working with F instead of $\mathbb{Q}$ plays little role in what follows, so let us assume for simplicity $F = \mathbb{Q}$ in this introduction, and also $\chi = 1$ for the same reason.

For general n, a given automorphic form f has at most $n!$ refinements f_i, and generically exactly $n!$ refinements. Moreover the Eigenvariety $\mathcal{E}$ has equidimension n, $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$

has dimension at least (but conjecturally exactly) $\frac{n(n+1)}{2}$ and there is a map $\mathcal{E}(\overline{\rho})^{(1)} \longrightarrow \mathcal{X}_{\overline{\rho}}^{pol}$ which forgets the refinement.

Definition 1.5. — The image of the map $\mathcal{E}(\overline{\rho}) \longrightarrow \mathcal{X}_{\overline{\rho}}^{pol}$ is called the *infinite fern* and is denoted by $\mathcal{F}(\overline{\rho})$.

Actually we can make our main theorem more precise :

Theorem 1.6. — *Under the previous hypothesis, the Zariski closure of the infinite fern $\mathcal{F}(\overline{\rho})$ in $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ is a non-empty union of irreducible components, each of which are of dimension $[F:\mathbb{Q}]\frac{n(n+1)}{2}$.*

Let us comment the various hypothesis we made. Contrary to [**HMS22**], we don't need to assume $\overline{\rho}$ absolutely irreducible if we use Chenevier's determinants, which we do (see section 3). The hypothesis of being conveniently modular is necessary to expect infinite fern to be non-empty, and is in practice very close to the usual modularity hypothesis which is anyway necessary. The hypothesis on the splitting of primes above p is technical, and we hope to come back on this question soon. The last hypothesis on χ is also technical and should be possible to relax in parts.

Following the strategy of Chenevier, the main goal is to prove that for a Zariski dense set of points ρ in the infinite fern, the part of the tangent space at ρ in $\mathcal{X}_{\overline{\rho}}^{pol}$ coming from $\mathcal{E}$ has dimension at least $\frac{n(n+1)}{2}$. This will imply that the closure of the infinite fern $\mathcal{F}(\overline{\rho})$, has dimension at least $\frac{n(n+1)}{2}$. As, by construction, automorphic points are Zariski dense in $\mathcal{E}$, thus in $\mathcal{F}(\overline{\rho})$, this will prove that the Zariski closure of automorphic points has dimension at least $\frac{n(n+1)}{2}$. Thus to prove the assertion on the tangent space, we need to show that the sum of the images of the tangent spaces $T_{f_i}\mathcal{E}$ in $T_{\rho_f}\mathcal{X}_{\overline{\rho}}^{pol}$, for well chosen automorphic forms f, has large enough dimension. But, clearly, as soon as $n \geqslant 3$ it is not sufficient that these tangent spaces are pairwise transversal, and this is the main difficulty to extend the proof of Gouvêa and Mazur. To overcome this problem, Chenevier suggested a strategy which he applied successfully when $n = 3$ and which can be sketched as follows:

1. find a good Zariski dense subset D of the infinite fern $\mathcal{F}(\overline{\rho})$, the image of $\mathcal{E}$ in $\mathcal{X}_{\overline{\rho}}$;
2. show that the analog of the question on the tangent spaces of points in D but for local deformation rings is valid ;
3. show that the Global situation "embeds well" in the local situation, and thus gives the result.

For the first part, Chenevier suggested to look at automorphic points ρ which he calls *generic*: they are crystalline at p and all their refinements are *non-critical.* More precisely, if ρ is crystalline at a place $v \mid p$, its restriction ρ_v to a decomposition group at v is characterized by a n-dimensional vector space $V = D_{cris}(\rho|_{\mathrm{Gal}(\overline{E}_v/E_v)})$ with its Hodge-Tate filtration $\mathcal{F}_{HT}$ (a complete flag) and Frobenius operator φ. The refinements of ρ corresponds to the complete flags of V stable by φ. We say that a refinement of ρ is non critical it is opposite to $\mathcal{F}_{HT}$. Actually, Chenevier proved that the second step works for crystalline

(1) an open-closed subvariety of $\mathcal{E}$

points which have n *well-positioned* non-critical flags and call those points *weakly-generic*. He moreover proved that weakly generic points (with some extra but harmless conditions) are Zariski dense in the infinite fern when $n = 3$ and uses those as the subset D.

Concerning the second point, the weakly generic condition is used to carry an induction in the local situation and prove that tangent spaces of local, refined, deformations problem spawn the tangent space of the full local deformation ring. This is where the definition of *well positioned* refinements comes from.

For the last point, actually it is enough to embed the situation at the level of tangent spaces. Chenevier proves that for all preimages of points in D, the map $\mathcal{E} \longrightarrow \mathcal{W}$ is etale, and deduces that some Selmer group vanishes at those points, allowing to *embed* infinitesimally the global situation into the local one, transversally to the cristalline locus, and thus deduce the result. This last argument is classical in the Taylor-Wiles method.

The main issue to generalise Chenevier's strategy in higher dimensions is that it is completely unclear that weakly generic points satisfies some density assumptions when $n \geqslant 4$ (see remark 5.2).

The strategy of [**HMS22**] is different but shares some similarities : for the first point they choose points which are crystalline with some genericity assumption[(2)] which are less restrictive than being *generic* or *weakly-generic*. Their set D is then automatically Zariski dense. For the second point, they use a local model for the local deformations spaces, which is of purely geometric nature, and a rather evolved but completely elementary argument allows to conclude in the second point, using not only all refinements but also *companion points*, which are extra-points appearing when the refinement is critical (whose existence is proved in [**BHS19**]).

Then the third point is the most delicate one and is proved by Talylor-Wiles-Kisin method via "patched eigenvarieties" (see [**BHS19**]) under restrictive Taylor-Wiles conditions.

In this article we use a strategy closer to Chenevier's, but using the local model of [**BHS19**] as in [**HMS22**]. Namely, using the local model and a careful study of its geometry, we first prove the second point without using companion points but rather generalizing Chenevier's transversality result at critical refinements (see section 7). For the first point, we show that setting for D the set crystalline points satisfying genericity conditions as in [**HMS22**] and which have moreover *enormous* image are actually Zariski dense in the infinite fern ; we call those points *almost-generic* (see Definition 5.4) because they will replace Chenevier's generic points in our argument. The density of these points is far from being automatic and the argument is originally due to Bellaïche-Chenevier and Taïbi (see section 5). Then, for the third point, we show that using the enormous image, and a result of Newton-Thorne, we have the vanishing of the expected Selmer group at points of D. We then show that this can be used to relate the global situation to the local one. As a byproduct, we obtain that our Eigenvariety is smooth at those points, as it was the case in other situations (see [**BHS19**, **Ber20**]) (see section 8). Then, a local calculation which was previously carried out in [**All16**], we show that our almost generic points are smooth points of $\mathcal{X}_{\bar{\rho}}^{pol}$ of the expected dimension.

(2) precisely on the Frobenius eigenvalues and Hodge-Tate weights

The results of Chenevier were combined with those by Allen ([**All19**]) (who proved that under some hypothesis every component contains an automorphic point) to prove full density of the infinite fern when $n \leqslant 3$, without assuming unobstructedness. We can adapt this generalisation also here,

***Corollary 1.7* (Allen). —** *Assume hypothesis 8.12, then the infinite fern is Zariski dense in* $\mathcal{X}_{\bar{\rho}}^{\chi-pol}$.

The only thing we need to care about for this corollary is that we use classical points which are automorphic representations for a similitude unitary group, which moreover contributes to the coherent H^0, whereas Allen's proof a priori only construct an (essentially) polarised autormorphic representation of GL_n.

Remerciements : We would like to warmly thank Gaetan Chenevier for very helpful suggestions concerning this work. We would also like to thank George Boxer, Laurent Clozel and Olivier Schiffmann for many interresting discussions. Finally, we would like to thank Anne vaugon and Marc Mezzarobba for their help for using SAGE to compute the local tangent spaces.

2. A remark on signs

By a theorem of Artin, all elements of order 2 are conjugate in $G_{\mathbb{Q}}$. Let $C_\infty \subset G_{\mathbb{Q}}$ be their conjugacy class and let $H \subset G_{\mathbb{Q}}$ be the closed subgroup generated by C_∞. There is a unique continuous morphism $\varepsilon : H \to \{\pm 1\}$ such that $\varepsilon(c) = -1$ for all $c \in C_\infty$. Let K be a number field. Then K is totally real if and only if $H \subset G_K$. Let E be a CM field with totally real subfield F, we have $H_1 := \ker \varepsilon \subset G_E$ and $G_F = G_E H$. Let $c \in C_\infty$. We can consider the action of c on G_E by conjugacy, and we have $G_F = G_E \rtimes \{1, c\}$. If ρ is a morphism of G_E in some group and $c \in C_\infty$, we set $\rho^c = \rho(c^{-1} \cdot c) = \rho(c \cdot c)$. The c-conjugacy induces an automorphism c of G_E^{ab}. As $H_1 \subset G_E$, this automorphism does not depend on the choice of $c \in C_\infty$. Let G be a finite abelian group.

***Lemma 2.1.* —** *Let $\chi : G_F \to G$ be a continuous character. Then $\chi|_{G_E} = (\chi|_{G_E})^c$. Moreover there exists a character $\psi : G_E \to G$ such that $\psi\psi^c = \chi|_{G_E}$ if and only if the elements $\chi(c)$ for $c \in C_\infty$ does not depend on the choice of $c \in C_\infty$.*

Proof. — Assume that the elements $\chi(c)$ for $c \in C_\infty$ does not depend on the choice of $c \in C_\infty$. After composition with the Artin map $\mathbb{A}_F^\times / F^\times \to G_F^{\mathrm{ab}}$, we can view χ as a morphism $\mathbb{A}_F^\times / F^\times \to G$ which is trivial on $\overline{F^\times (F_\infty^\times)^\circ}$. We have to prove that there exists a character $\psi : \mathbb{A}_E^\times / E^\times \to G$ which is trivial on $\overline{E^\times E_\infty^\times}$ and such that $\psi \circ N_{E/F} = \chi \circ N_{E/F}$. But the same proof that in [**CHT08**, Lem. 4.1.4] shows that χ is trivial on $N_{E/F}(\mathbb{A}_E^\times) \cap \overline{E^\times E_\infty^\times}$.

Conversely assume that $\chi|_{G_E} = \psi\psi^c$ for some ψ and let c_1 and c_2 be two elements of C_∞. Then

$$\begin{aligned}\chi(c_1 c_2^{-1}) &= \chi(c_1 c_2) = \psi(c_1 c^2 c_2)\psi(c c_1 c_2 c) \\ &= \psi(c_1 c)\psi(c c_2)\psi(c c_1)\psi(c_2 c) = \psi^{-1}(c c_1)\psi(c c_1)\psi(c c_2)\psi^{-1}(c c_2) = 1. \quad \square\end{aligned}$$

Note that a continuous morphism $\chi : G_E \to G$ extends to a morphism $G_F \to G$ if and only if $\chi = \chi^c$. The fact that the extension of χ to G_F satisfies the assumptions of Lemma 2.1 depends only on the restriction of χ to G_E and is equivalent to the fact that $\chi|_{H_1}$ is trivial. Lemma 2.1 can be reformulated as follows :

Corollary 2.2. — *Let $\chi : G_E \to G$ be a continuous morphism from G_E to a finite abelian group such that $\chi = \chi^c$. There exists a continuous morphism $\psi : G_E \to G$ such that $\psi\psi^c = \chi$ if and only if $\chi|_{H_1}$ is trivial. Note that this condition is automatically satisfied if the cardinal of G is odd.*

In this article we will be interested in the density of modular point in deformation rings. We will say that $\rho : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$, a (semi-simple) Galois representation, is *GL_n-modular*[3] if there exists a cuspidal, essentially conjugate self-dual (regular algebraic) automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\rho \simeq \rho_{\Pi,p}$; where $\rho_{\Pi,p}$ is associated in the sense of [**BGGT14**].

Let $\chi : G_E \to \overline{\mathbb{Q}_p}^\times$ be a continuous character and $\rho : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ a continuous representation such that ρ is polarised by χ, which means

$$\rho^\vee \simeq \rho^c \otimes \chi\varepsilon^{n-1}. \tag{1}$$

If ρ is irreducible then we can define its sign, with respect to $\chi\varepsilon^{n-1}$, $\lambda \in \{\pm 1\}$ as in [**BC11**] 1.1. This is the sign of the pairing appearing in (1).

***Theorem 2.3* (Bellaïche-Chenevier).** — *Let Π be a cuspidal automorphic representation of $GL_n(E)$ that is conjugate self-dual, and regular algebraic. Let ρ_Π^u be the associated Galois representation as in Corollary A.8, and $\rho = \rho_\Pi^u\psi$ for some character $\psi : G_E \longrightarrow \overline{\mathbb{Q}_p}$. Then every irreducible constituent r of ρ satisfying (1) has sign $\lambda = +1$ with respect to $(\psi\psi^c)^{-1}\varepsilon^{n-1}$.*

Proof. — If $\psi = 1$ this is [**BC11**] Theorem 1.2. In general $\rho = \rho_\Pi^u\psi$ and $\chi := (\psi\psi^c)^{-1}$ and ρ_Π^u has sign $+1$ for ε^{n-1} thus ρ (and all its irreducible factors) have sign $+1$ with respect to $\chi\varepsilon^{n-1}$ by [**BC11**] Lemma 2.1. □

3. Deformation spaces

Denote by k a topological field and $\mathcal{O}$ a complete noetherian local $\mathbb{Z}_p$-algebra with residue field k.

Fix $E/\mathbb{Q}$ a totally imaginary CM-extension of number fields with maximal totally real subfield F, and fix S a finite set of finite places of E containing the places above p, and the ramified places of E, and denote

$$G_{E,S} = \mathrm{Gal}(E_S/E),$$

the Galois group of the maximal unramified outside S extension of E.

Suppose given

$$\overline{\rho} : G_{E,S} \longrightarrow \mathrm{GL}_n(k),$$

[3] We say GL_n-modular to distinguish from the rest of the text where we will get modular points using (similitude) unitary groups so in Definition 4.1 we give a slightly different notion, which we call just modular. Of course by base change (see Appendix) modular is a particular case of GL_n-modular.

a continuous *semi-simple* Galois representation. From now on we choose $c \in G_F \backslash G_E$ such that $c^2 = 1$ a complex conjugation, denoting similarly its image in $\mathrm{Gal}(E/F)$. As $\mathcal{O}$ is a $\mathbb{Z}_p$-algebra, we have a map $\mathbb{Z}_p \longrightarrow k$, thus we can see $\overline{\varepsilon}$ as the cyclotomic character with values in k. We also assume that $\overline{\rho}$ is polarized by $\overline{\chi}$, i.e.

$$\overline{\rho}^\vee \simeq \overline{\rho}^c \otimes (\overline{\varepsilon}^{n-1}\overline{\chi}),$$

for some character $\overline{\chi} : G_{E,S} \longrightarrow k^\times$ satisfying $\overline{\chi}^c = \overline{\chi}$. Following **[CHT08]** we introduce

$$\mathcal{G}_n = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \mathrm{Gal}(E/F),$$

where $c \in \mathrm{Gal}(E/F)$ acts on $(g, x) \in \mathrm{GL}_n \times \mathrm{GL}_1$ via $(x^t g^{-1}, x)$. We denote ν the homomorphism $\mathcal{G}_n \longrightarrow \mathrm{GL}_1$ sending (g, x) to x and c to -1. Finally we denote $\mathcal{C}_{\mathcal{O}}$, or $\mathcal{C}$ if the context is clear, the category of artinian local $\mathcal{O}$-algebras with residue field k.

Hypothesis 3.1. — From now on in this section, we suppose that we are in either one of the two situations : $k \subset \overline{\mathbb{F}_p}$ with discrete topology, $\mathcal{O}$ a finite totally ramified extension of $W(k)$, or $k \subset \overline{\mathbb{Q}_p}$ a finite extension of $\mathbb{Q}_p$ with its p-adic topology and in this case we set $\mathcal{O} = k$. In the second case $\overline{\varepsilon} = \varepsilon$ is just the $\mathbb{Z}_p^\times$-valued cyclotomic character.

Denote by $\mathrm{tr}\,\overline{\rho}$ the Determinant (in the sense of Chenevier [**Che14**, Définition 1.5]) of $\overline{\rho}$. As $\overline{\rho}$ is semi-simple it is completely determined by $\mathrm{tr}\,\overline{\rho}$ (by [**Che14**, Cor. 2.13]). We fix once and for all a continuous character $\chi : G_{E,S} \longrightarrow \mathcal{O}^\times$ a lifting $\overline{\chi}$ and such that $\chi^c = \chi$.

Definition 3.2. — We denote by $\mathcal{F}_{\overline{\rho}}^{\chi-pol}$ the functor that associate to any object A of $\mathcal{C}$ the set of continuous determinants D lifting $\mathrm{tr}\,\overline{\rho}$ such that $D^\vee = D^c \otimes \chi\varepsilon^{n-1}$. It is pro-representable by a ring $R_{\overline{\rho}}^{\chi-pol}$ ([**Che14**, Prop. 3.3][(4)]). We denote the associated formal scheme $\mathfrak{X}_{\overline{\rho}}^{\chi-pol} = \mathrm{Spf}(R_{\overline{\rho}}^{\chi-pol})$. When k is a finite field of characteristic p, we denote the generic fibre of $R_{\overline{\rho}}^{\chi-pol}$ by

$$\mathcal{X}_{\overline{\rho}}^{\chi-pol} := \mathrm{Spf}(R_{\overline{\rho}}^{\chi-pol})^{rig}.$$

If $\overline{\rho}$ is absolutely irreducible, this coincides with the rigid fiber of the polarized-by-χ deformation space of $\overline{\rho}$.

Our goal is to understand the geometry of modular points in $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ when $k \subset \overline{\mathbb{F}_p}$, $\mathcal{O} = \mathcal{O}_K$, $K/\mathbb{Q}_p$ finite. Keep the slightly greater generality for now and assume that $\overline{\rho}$ is Schur.[(5)] Denote its sign λ. Then we can extend χ, which satisfies $\chi^c = \chi$ to $G_F \simeq G_E \rtimes \mathrm{Gal}(E/F)$ by setting $\chi(c) = \pm 1$. We set $\chi(c) := (-1)^n\lambda$ and $\overline{\chi}(c) := (-1)^n\lambda$, so that $\mu := \chi\varepsilon^{n-1}$ satisfies $\mu(c) = -\lambda$. By **[CHT08]** Lemma 2.1.1 we can thus extend $\overline{\rho}$ to a continuous

$$\overline{r} : G_{F,S} \longrightarrow \mathcal{G}_n(k),$$

such that $c \in G_F$ is sent to $c \in \mathrm{Gal}(E/F)$ via $\overline{r}$ and projection and $\nu \circ \overline{r} = \overline{\chi}^{-1}\overline{\varepsilon}^{1-n}$ (as extended before to G_F).

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(4) for $R_{\overline{\rho}}$, and then $R_{\overline{\rho}}^{\chi-pol} = R_{\overline{\rho}}/I$ with $I = (D^{univ,\vee}(g) - D^{univ,c}(g)\chi\varepsilon^{n-1}(g), g \in G)$

(5) We define it the following way : choose $\overline{r} : G_F \longrightarrow \mathcal{G}_n(k)$ extending $\overline{\rho}$ by **[CHT08]** Lemma 2.1.1. Then we say that $\overline{\rho}$ is Schur if $\overline{r}$ is. By **[CHT08]** Lemma 2.1.7 this is independant of $\overline{r}$. In particular it is satisfied if $\overline{\rho}$ is absolutely irreducible

Definition 3.3. — Let $\mathrm{Def}_{\overline{r}}^{\chi}$ be the functor that associates to any object R of $\mathcal{C}$ the set $\mathrm{Def}_{\overline{r}}^{\chi}(R)$ of lifts $r : G_{F,S} \to \mathcal{G}_n(R)$ of $\overline{r}$ such that $\nu \circ r = \overline{\chi}^{-1}\overline{\varepsilon}^{1-n}$ considered up to $1 + \mathfrak{m}_R M_n(R)$-conjugation. As in [**CHT08**, Prop. 2.2.9][(6)], this functor is pro-represented by a local complete noetherian $\mathcal{O}$-algebra $R_{\overline{r}}^{\chi}$. When k is a finite field of characteristic p, we denote by $\mathcal{X}_{\overline{r}}^{\chi}$ the generic fiber of the formal scheme $\mathfrak{X}_{\overline{r}}^{\chi} = \mathrm{Spf}(R_{\overline{r}}^{\chi})$.

In the following, all cohomology groups are continuous cohomology groups.

Proposition 3.4. — *Assume that $\overline{\rho}$ is Schur,* $\mathrm{char}\, k \neq 2$, $\chi(c') = (-1)^n$ *for all complex conjugacy c' and, if k is of characteristic p, $R_{\overline{r}}^{\chi}[1/p] \neq 0$. Then*

$$\dim(R_{\overline{r}}^{\chi}[1/p]) \geqslant \dim_k H^1(G_{F,S}, \mathrm{ad}(\overline{r})) - \dim_k H^2(G_{F,S}, \mathrm{ad}(\overline{r})) = \frac{n(n+1)}{2}[F : \mathbb{Q}].$$

Moreover the topological $\mathcal{O}[1/p]$-algebra $R_{\overline{r}}^{\chi}[1/p]$ is formally smooth of relative dimension $\frac{n(n+1)}{2}[F : \mathbb{Q}]$ if $H^2(G_{F,S}, \mathrm{ad}(\overline{r})) = 0$.

Proof. — This appeared already in [**CHT08**, **All16**], let us give the argument. As $\overline{\rho}$ is Schur, $\mathrm{ad}(\overline{r})^{G_F} = H^0(G_F, \mathrm{ad}(\overline{r})) = 0$ by [**CHT08**, Lem. 2.1.7(3)]. For each place $v|\infty$, we have ([**CHT08**, Lem. 2.1.3])

$$\dim_k H^0(G_{F_v}, \mathrm{ad}(\overline{r})) = \frac{n(n + \overline{\chi}\overline{\varepsilon}^{n-1}(c_v))}{2} = \frac{n(n-1)}{2}.$$

Now the equality

$$\dim_k H^1(G_{F,S}, \mathrm{ad}(\overline{r})) - \dim_k H^2(G_{F,S}, \mathrm{ad}(\overline{r})) = \frac{n(n+1)}{2}[F : \mathbb{Q}]$$

follows from [**CHT08**, Lem. 2.3.3][(7)] when k is a finite field and from [**All16**, Lem. 1.3.4] when k is a finite extension of $\mathbb{Q}_p$.

When k is a finite extension of $\mathbb{Q}_p$, the result follows from using the analogue of [**CHT08**, Cor. 2.2.12] (but without the $+1$ since here $\mathcal{O} = k$). When k is a finite field, it follows from [**CHT08**, Cor. 2.2.12] that

$$\dim(R_{\overline{r}}^{\chi}) \geqslant 1 + \frac{n(n+1)}{2}[F : \mathbb{Q}].$$

Let $x \in \mathrm{Spec}(R_{\overline{r}}^{\chi}[1/p])$ be a closed point, $\mathfrak{p}_x$ the corresponding prime ideal and $r_x : G_F \to \mathcal{G}_n(k(x))$ the corresponding representation. It follows from [**All16**, Prop. 1.3.11(1)] that the localization-completion of $R_{\overline{r}}^{\chi}$ at $\mathfrak{p}_x$ is isomorphic to $R_{r_x}^{\chi}$. It follows that

$$\dim(R_{\overline{r}}^{\chi}[1/p]) \geqslant \dim(R_{r_x}^{\chi}) \geqslant \frac{n(n+1)}{2}[F : \mathbb{Q}]$$

using the case where k has characteristic 0.

The assertion concerning the formal smoothness follows from [**CHT08**, Cor. 2.2.12] and [**All16**, Prop. 1.3.11] □

(6) There the field k is finite, but we can check that everything carries over in our setting, as already remarked in [**Kis09**]

(7) Note that in [**CHT08**, 2.3], it is supposed that the places of S are split in E but this is not used in their Lemma 2.3.3.

Remark 3.5. — The hypothesis on the sign of χ will be satisfied in the rest of the text where we choose $\chi = \psi_0\psi_0^c$, actually because of the sign theorem 2.3. Indeed, by **[CHT08]** Lemma 2.1.1 we need to extend $\mu = \chi\varepsilon^{n-1}$ to $G_{F,S}$, for an absolutely irreducible ρ, by sending c to $-\lambda$. Thus the previous hypothesis is equivalent to $\lambda = 1$. Here we use crucially that χ is of the form $\psi_0\psi_0^c$ as $\overline{\rho}$ will be of the form $\overline{\rho_\Pi^u}\psi_0$. Actually, if we keep track of the L-parameter of the similitude character of π, an automorphic representation of GU, we have in most case a natural χ of this form which depends on π. In this article we forget this similitude parameter and force polarisation by $\chi\varepsilon^{n-1}$ using ψ_0. We will use the previous proposition in the case of $k \subset \overline{\mathbb{Q}_p}$ and $\mathcal{O} = \mathbb{Z}_p$ at specific rigid points of $\mathfrak{X}_{\overline{\rho}}^{\chi-pol}$ in Theorem 8.9.

Proposition 3.6. — *Denote $\overline{\rho}$ as before. Suppose it is absolutely irreducible, and denote $\overline{r}$ the chosen $\mathcal{G}_n$-extension as before. Suppose* $\mathrm{Char}(k) \neq 2$. *Then the natural map*

$$R_{\overline{r}}^{\chi} \longrightarrow R_{\overline{\rho}}^{\chi-pol}$$

is an isomorphism.

Proof. — This is also [**All19**, Prop. 2.2.3]. Denote ρ, ρ' resp. R, R' valued points of $\mathcal{F}_{\overline{\rho}}^{\chi-pol}$, with $R' \twoheadrightarrow R$, and $\rho' \otimes R = \rho$. Suppose we have a fixed pairing $<,>: \rho \otimes \rho^c \longrightarrow \chi^{-1}\varepsilon^{1-n}$ inducing $r : G_{E,S} \longrightarrow \mathcal{G}_n(R)$ by **[CHT08]** Lemma 2.1.1. Choose any pairing fixing $<,>_0'$ for ρ'. Then reducing to R this gives a pairing for ρ, but as $\overline{\rho}$ is absolutely irreducible, ρ is also and thus there is only one pairing up to scalar for ρ, i.e. $<,>_0' \otimes R = \alpha <,>$ for some α in $R^\times$. Choose a lift β of α^{-1}, then set $<,>' := \beta <,>_0'$, then $<,>'$ reduces to $<,>$ and to $(\rho', \chi, <,>')$ is associated by **[CHT08]** Lemma 2.1.1 an $r' : G_{E,S} \longrightarrow \mathcal{G}_n(R')$, reducing to r. Let r'' another point over R', inducing ρ' and reducing to r , then it corresponds to $\gamma < .,. >'$ with $\gamma \equiv 1 \pmod{m_{R'}}$, thus writing $\gamma = 1 + m$ with $m \in m_{R'}$ we have $\gamma = (1 + \frac{1}{2}m)^2 \pmod{m_{R'}^2}$ and as R' is artinian, a direct induction shows that γ is a square in R', thus $r' = r''$.[(8)]

The same argument with $\overline{r}$ for r and r for r' shows that we can actually choose r inside $\mathfrak{X}_{\overline{r}}^{\chi}$ (and thus automatically for any r' above) and thus proves etaleness, and surjectivity. As the map is an isomorphism in special fiber, this is an isomorphism. □

We will need to assume the following technical hypothesis in this article.

Hypothesis 3.7. — Assume $\chi : G_{E,S} \to \overline{\mathbb{Q}_p}^\times$ is a continuous character crystalline at p. We assume also that $\chi = \chi^c$ and that $\chi|_{H_1}$ is trivial.

From now on we fix an isomorphism $\iota : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{C}$. Let $\psi : \mathbb{A}_F^\times / F^\times \to \mathbb{C}^\times$ be the unique character such that $\psi_v = \iota \circ \chi_v \circ \mathrm{Art}_{F_v}$ for all $v \nmid p$, where $\mathrm{Art}_{F_v} : F_v^\times \to G_{F_v}^{\mathrm{ab}}$ is the local reciprocity map. We say that an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ is ***polarized by*** ψ if it is regular algebraic cuspidal and such that $\Pi^c \simeq \Pi^\vee \otimes (N_{E/F} \circ \psi)$. We recall that if Π is a polarized by ψ automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$, there exists a unique continuous semisimple Galois representation

$$\rho_{\Pi,\iota} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$$

(8) **[CHT08]** is written over a field, but their proof applies here

satisfying the conditions of [**BGGT14**, Thm. 2.1].

Definition 3.8. — A point of $x \in \mathcal{X}_{\overline{\rho}}^{\chi-pol}(\overline{\mathbb{Q}_p})$ is *GL_n-modular*[9] if there exists a polarized by ψ automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\rho_{\Pi,\iota} \simeq \rho_x$.

Our goal is to prove a density result for automorphic points in the previous (polarized) deformation rings. It follows from [**CHT08**, Lem.4.1.5] that there exists a continuous character $\psi_0 : G_{E,S} \longrightarrow \overline{\mathbb{Q}_p}^{\times}$ which is crystalline at p and such that $\chi = \psi_0\psi_0^c$. We fix such a character ψ_0.

We start with the following reduction to lighten slightly the notations in the rest of the text.

Lemma 3.9. — *Assume χ is as before (in particular crystalline at p), and $\overline{\rho}$ satisfies $\overline{\rho}^\vee \simeq \overline{\rho}^c \otimes \varepsilon^{n-1}$. Then there is an isomorphism, which identifies modular points,*

$$\mathcal{X}_{\overline{\rho}}^{1-pol} \xrightarrow{\sim} \mathcal{X}_{\overline{\rho}\psi_0^{-1}}^{\chi-pol}.$$

In particular it is enough to prove theorem 1.6 for $\chi = 1$.

Proof. — Indeed let $\psi_0 : G_{E,S} \longrightarrow \overline{\mathbb{Q}_p^\times}$ the crystalline character given by [**CHT08**] Lemma 4.1.5. Then ψ_0 is automorphic. Moreover, the isomorphism is given by

$$\rho \longmapsto \rho\psi_0^{-1}.$$

This is obviously an isomorphism, and because ψ_0 is automorphic it identifies (GL_n-)modular points on both sides. □

4. Eigenvarieties and the infinite fern

The are at least two ways to define Eigenvarieties as explained in [**BC09**], which – at least – in our case of interest end up to be the same. We will need to assume some technical hypothesis, see Hypothesis 4.2, on top of the assumption on χ (see Hypothesis 3.7). By the previous lemma, we can assume $\chi = 1$.

Let $\overline{\rho}$ be a semi-simple, polarised-by-ε^{n-1} as before and $\mathfrak{X}_{\overline{\rho}}^{pol} := \mathfrak{X}_{\overline{\rho}}^{1-pol}$ its polarised pseudodeformation space. Let G be the quasi-split similitude unitary group of dimension n over $\mathbb{Q}$ whose R-points, for R a $\mathbb{Q}$-algebra, are:

$$G(R) = \{(g,\nu) \in \mathrm{GL}_n(R \otimes_{\mathbb{Q}} E) \times R^\times \mid {}^t c(g)Jg = \nu J\}$$

where J is the $n \times n$ matrix $\begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}$. Moreover let G_1 be the kernel of the morphism $\nu : G \to \mathbb{G}_m$. As p is unramified in E, we also fix a reductive model $G_{\mathbb{Z}_p}$ of G defined by the similar formula (replacing $R \otimes_{\mathbb{Q}} E$ by $R \otimes_{\mathbb{Z}} \mathcal{O}_E$).

We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ that we use to identify the embeddings of E (resp. F) in $\overline{\mathbb{Q}}_p$ with the set Σ_E (resp. Σ_F) of embeddings of E (resp. F) in $\mathbb{C}$. We fix a CM type Φ for E. For $\sigma \in \Sigma_E$, we use the notation $\overline{\sigma} = \sigma \circ c$. If $\tau \in \Sigma_F$, let $\sigma_\tau \in \Sigma_E$ be the unique element such that $\tau = \sigma_\tau|_F$ and $\sigma_\tau \in \Phi$.

---

(9) Compare with Definition 4.1.

We fix a PEL datum $(E, c, V, \langle\cdot,\cdot\rangle, h)$ for the previous group G and denote its signature $(p_{\sigma_\tau}, q_{\sigma_\tau})_{\tau\in\Sigma_F}$ at infinity[(10)]. In particular we have that $p_{\sigma_\tau} + q_{\sigma_\tau} = n$ doesn't depend on τ. We define more generally $(p_\sigma)_{\sigma\in\Sigma_E}$ by $p_\sigma = p_{\sigma_\tau}$ if $\sigma = \sigma_\tau$ and $p_\sigma = q_{\sigma_\tau} = n - p_{\sigma_\tau}$ if $\sigma = \overline{\sigma_\tau}$. We also sometimes abuse notation and write p_τ, q_τ for $p_{\sigma_\tau}, q_{\sigma_\tau}$. Let (G, h) be a Shimura datum associated to G. We let $\mathcal{S} = (S_K)_K$ be the tower of Shimura varieties for (G, h) ([**Lan13**] or [**Her19**] which we will use later). Let $\mu : \mathbb{G}_m \to G_{\mathbb{C}}$ be the cocharacter associated to h and let P be the parabolic subgroup fixing the Hodge filtration associated to μ. Let M be the Levi subgroup of P fixing the Hodge decomposition of $V_{\mathbb{C}}$ (defined over some extension L of the reflex field). Let $\mathfrak{p}$ be the Lie algebra of P and let K_∞ be the centralizer of $h(i)$ in $G_1(\mathbb{R})$.

Definition 4.1. — We say that a polarised-by-ε^{n-1} representation

$$\rho : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p}),$$

is ***modular*** if ρ is (strongly essentially) associated to a cuspidal algebraic automorphic representation π for G as in Definition A.2. We say that ρ is ***holomorphically modular*** if its Hecke eigensystem appears in the space of cuspidal sections of some coherent automorphic sheaf on some Shimura variety of $\mathcal{S}$. This is equivalent to the fact that π is cuspidal and holomorphic at infinity; i.e. $H^0(\mathfrak{p}, K_\infty, \pi_\infty \otimes \sigma) \neq 0$ for some algebraic representation σ of K_∞ [(11)] by [**Har90b**] Proposition 5.4.2.

We say that $\overline{\rho}$ is modular if it admits a lift ρ which is modular. We say that $\overline{\rho}$ is ***conveniently modular*** if it has a lift ρ associated to a cuspidal automorphic representation π which can be chosen unramified at p and outside S and its Hecke eigensystem appears in i-th interior coherent cohomology group on some Shimura variety of $\mathcal{S}$, with values in some coherent automorphic sheaf, for some $i \geqslant 0$.

If K^p is a compact open open subgroup of $G(\mathbb{A}^{p,\infty})$, we say that $\overline{\rho}$ is ***conveniently modular of tame level*** K^p if π can be moreover chosen such that $\pi^{K^p} \neq 0$.

Hypothesis 4.2. — For the rest of the article, we assume that every $v|p$ in F is unramified, and splits in E. Moreover we assume that $\overline{\rho}$ is conveniently modular.

In particular, if v is a place of F dividing p, among the two places $w, \overline{w}$ of E above v only one, say w, corresponds to an element of Φ. We fix this choice, which allows us to identify E_w with F_v. Choose a sufficiently large p-adic field L such that M and P are defined over L and L splits E, i.e. $E \otimes_{\mathbb{Q}_p} L = \prod_{w|p\in E} L$. Let $\mathcal{T}$ be the rigid space over L given by $\prod_{v|p} \mathrm{Hom}((F_v^\times)^n, \mathbb{G}_m)$, and $\mathcal{W} = \prod_{v|p} \mathrm{Hom}((\mathcal{O}_{F_v}^\times)^n, \mathbb{G}_m)$ the weight space. There is thus a restriction map

$$\mathcal{T} \longrightarrow \mathcal{W}.$$

We fix a tame level outside of p, K^p, which is hyperspecial outside S and deep enough so that $\overline{\rho}$ is conveniently modular of tame level K^p.

(10) Because G is quasi-split these integers are explicit, but we keep the slightly general notation as we think it is a bit clearer.

(11) These hypothesis are here to assure a concrete (= computable) way to verify if our π "appears" in an Eigenvariety. We could introduce the notion of ***p-adically modular*** for which we ask for a Hecke eigensystem appearing in the considered Eigenvariety $\mathcal{E}$ whose associated trace is $\mathrm{tr}\,\rho$. It is enough to assume p-adic modularity for $\overline{\rho}$ to get Theorem 1.6

Let $\mathcal{Z}'_{K^p} \subset \mathfrak{X}_{\overline{\rho}}^{\chi-pol}(\overline{\mathbb{Q}_p}) \times \mathcal{T}(\overline{\mathbb{Q}_p})$ the set of pairs (D,δ) where D is the determinant associated to (the Galois representation of) a cuspidal, regular, algebraic, unramified at p automorphic form Π for G appearing in degree 0 coherent cohomology [12] by Corollary A.8, of level K^p outside p, of Hodge-Tate weights $k_{v,\tau,1} > k_{v,\tau,2} > \cdots > k_{v,\tau,n}$[13] for each $v|p$ in F,τ, and δ such that for all v,i, $\delta_{v,\tau,i}$ coincides on $\mathcal{O}_{F_v}^\times$ with $\prod_\tau \tau^{k_{v,\tau,i}}$ and sends p to $\phi_{v,i}$, where $\phi_{v,1},\dots,\phi_{v,n}$ is an admissible refinement for Π_v (and obviously such that D lifts $\overline{\rho}$).[14]

Remark 4.3. — By [**Box15**] Theorem D, [**PS16**] or [**GK15**] Theorem I.3.1, we have that, under the hypothesis 4.2, $\mathcal{Z}'_{K^p}$ is non empty, i.e. we can choose a lift of $\overline{\rho}$ that is holomorphically modular.

Definition 4.4. — The Eigenvariety for G, $\overline{\rho}$, $\chi = 1$ and K^p is the Zariski closure

$$\mathcal{E}_{K^p}(\overline{\rho}) \subset \mathcal{X}_{\overline{\rho}}^{pol} \times \mathcal{T},$$

of $\mathcal{Z}'_{K^p}$. The infinite fern $\mathcal{F}_{K^p}(\overline{\rho})$ is the image of $\mathcal{E}_{K^p}(\overline{\rho})$ by the first projection.

As G is a unitary similitude group with similitude in $\mathbb{Q}$, thus giving rise to a PEL Shimura datum, and p is unramified in E, we also constructed in [**Her19**] an Eigenvariety for G, for any type K^p outside p. Actually these two constructions compare, and allow us to deduce the following proposition.

Remark 4.5. — Actually we could take G any similitude unitary group with similitude factor in $\mathbb{Q}$ instead of the quasi-split one. Indeed, as long as p is unramified for G the construction of [**Her19**] applies and we get the following proposition. In particular, if we have a result analogous to [**Her19**] for ramified primes (i.e. for primes $v|p$ in F which are ramified, but still assuming $v = w\overline{w}$ in E) then all the methods of this article applies (see [**BP20**]). For the moment, we still need our p-adic group to be (a product of) GL_n to use results on the trianguline variety, but we hope to come back on this question in the future.

Proposition 4.6. — *The rigid space $\mathcal{E}_{K^p}(\overline{\rho})$ is equidimensional of dimension $n[F:\mathbb{Q}]$. The map*

$$h : \mathcal{E}_{K^p}(\overline{\rho}) \longrightarrow \mathcal{W},$$

is locally, on the goal and the source, finite. In particular the image of any irreducible component of $\mathcal{E}_{K^p}(\overline{\rho})$ is open in $\mathcal{W}$. Moreover there exists, for all $C > 0$, $\mathcal{Z}_C \subset \mathcal{Z}'_{K^p}$ consisting of classical points, crystalline at p, which are moreover C-very regular (i.e. its Hodge-Tate weights satisfies $k_{v,\tau,i} > k_{v,\tau,i+1} + C$ for all v,τ,i) which is Zariski dense and accumulation at every point of $\mathcal{Z}'_{K^p}$.

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(12) this means that $H^0(\mathfrak{p}, K_\infty, \pi_\infty \otimes V) \neq 0$ for a finite dimensionnal representation of K_∞. In particular D is holomorphically modular.

(13) We choose the convention for which the cyclotomic character has Hodge-Tate weight +1

(14) for these local data at p we have used the implicit choice of $w|v$

We need to introduce a few notations. Let T be the diagonal torus of $G_{\mathbb{Z}_p}$. Its group of $\mathbb{Q}_p$-points has the following description,

$$T(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a_{1,w} & & \\ & \ddots & \\ & & a_{n,w} \end{pmatrix} \in \prod_{w|p \text{ in} E} \mathrm{GL}_n(E_w), \exists r \in \mathbb{Q}_p^\times, a_{i,w}a_{n-i+1,\overline{w}} = r, \forall i, w \right\}.$$

Denote by T^1 the subtorus with trivial similitude character (i.e. $r = 1$). We identify $\mathcal{T}$ with the space of characters of $T^1(\mathbb{Q}_p)$ using the isomorphism $(F_v^\times)^n \simeq T^1(\mathbb{Q}_p)$ sending $(a_{1,v}, \dots, a_{n,v})$ to the diagonal matrix of $\mathrm{GL}_n(E_w)$ with diagonal $(a_{1,v}, \dots, a_{n,v})$, via the identification $E_w \simeq F_v$ where $w \mid v$ and $w \in \Phi$. $T(\mathbb{Q}_p)$ (resp. $T^1(\mathbb{Q}_p)$) can be identified also with a subgroup of the L-points of the torus (resp. subtorus of $r = 1$ elements) of $M \simeq \mathbb{G}_m \times \prod_{v|p \text{ in } F} \prod_{\sigma_\tau \in \mathrm{Hom}(F_v, \overline{\mathbb{Q}_p})} \mathrm{GL}_{p_{\sigma_\tau}} \times \mathrm{GL}_{q_{\sigma_\tau}}$, using,

$$\begin{pmatrix} a_{w,1} & & \\ & \ddots & \\ & & a_{w,n} \end{pmatrix}_w \longmapsto \left(r, \left(\begin{pmatrix} \tau(a_{w,p_{\sigma_\tau}})^{-1} & & \\ & \ddots & \\ & & \tau(a_{w,1})^{-1} \end{pmatrix}, \begin{pmatrix} \overline{\tau}(a_{\overline{w},q_{\sigma_\tau}})^{-1} & & \\ & \ddots & \\ & & \overline{\tau}(a_{\overline{w},1})^{-1} \end{pmatrix} \right)_\tau \right)$$

Definition 4.7. — Let $\kappa = (k_{\sigma,i})_{\substack{\sigma \in \Sigma_E \\ 1 \leqslant i \leqslant p_\sigma}} \in \mathbb{Z}^{n[F:\mathbb{Q}]}$. We say that a character $\chi \in \mathcal{W}(\mathbb{C}_p)$ is *algebraic of coherent weight* κ if
(2)

$$\forall z = (z_{v,i}) \in \prod_{v|p} (\mathcal{O}_{F_v}^\times)^n, \quad \chi(z) = \prod_{v|p} \prod_{i=1}^{p_\tau = p_{\sigma_\tau}} \sigma_\tau(z_{v,i})^{k_{\sigma_\tau,i}} \prod_{i=1}^{q_\tau = p_{\overline{\sigma_\tau}}} \sigma_\tau(z_{n+1-i})^{-k_{\overline{\sigma_\tau},i}}.$$

We say that an algebraic character of coherent weight κ is *M-dominant* if $k_{\sigma,i} \geqslant k_{\sigma,i+1}$ for $\sigma \in \Sigma_E$ and $1 \leqslant i \leqslant p_\sigma - 1$.

This corresponds to the choice of the upper triangular Borel for M, in the sense that if we have character κ of M which is dominant for the upper Borel of M, then its restriction to $T^1(\mathbb{Q}_p)$ via the previous embedding gives a ***M-dominant*** χ in the previous sense. Suppose χ is algebraic for some coherent weight κ. For $h = (h_{\tau,i})_{\tau \in \Sigma_F, 1 \leqslant i \leqslant n} \in \mathbb{Z}^{n[F:\mathbb{Q}]}$, if[15]

$$\chi(z) = \prod_{\tau \in \Sigma_F} \sigma_\tau(z_i)^{h_{\tau,i}},$$

then we say that χ is of ***infinitesimal weight*** h. We say that such a χ is ***dominant*** (or ***G-dominant***) if $h_{\tau,1} \geqslant h_{\tau,2} \geqslant \cdots \geqslant h_{\tau,n}$ for all τ.

Proof. — Let $\mathcal{E}'$ together with a locally finite map $w : \mathcal{E}' \longrightarrow \mathcal{W}$ the Eigenvariety for G of tame level K^p constructed in **[Her19]**. It is an equidimensionnal rigid space of dimension $n[F : \mathbb{Q}]$.

Let S be a finite set of places of F containing the places dividing p and the places where K^p is not hyperspecial. For $v \notin S$, let $\mathcal{H}_v$ be the spherical Hecke algebra $\mathbb{Z}[G(F_v)//K_v^p]$ of G and let $\mathcal{H}^S = \bigotimes_{v \notin S} \mathcal{H}^v$. For $v|p$, let $\mathcal{A}_v$ be the (commutative) $\mathbb{Z}$-algebra generated by $T_n^-(F_v)/T_n(\mathcal{O}_{F_v})$ and their inverses with T_n the diagonal torus of GL_n, and $T_n^-(F_v)$ the

(15) This just means that $k_{\sigma_\tau,i} = h_{\tau,i}, i \leqslant p_\tau$ and $-k_{\overline{\sigma_\tau},i} = h_{\tau,n+1-i}$ for $i > p_\tau$.

subgroup of matrices $\mathrm{Diag}(a_1,\dots,a_n)$ with $v(a_i) \geqslant v(a_{i+1})$. Let $\mathcal{A}(p) = \bigotimes_{v|p}\mathcal{A}(v)$ be the Atkin-Lehner algebra. It follows from [**Her19**, §7][16] that there exists homomorphism $\lambda : \mathcal{H}^S \to \Gamma(\mathcal{E}', \mathcal{O}^+_{\mathcal{E}'})$ and $\mathcal{A}(p) \to \Gamma(\mathcal{E}', \mathcal{O}_{\mathcal{E}'})$, sending $\mathrm{Diag}(\underbrace{1,\dots,1}_{i \text{ times}}, p^{-1},\dots,p^{-1})_v$ to a Hecke operator $U_{v,i}$ at v, such that, if $z \in \mathcal{E}'(\mathbb{C}_p)$, the evaluation of these morphisms at z induce a non-zero eigenspace in $H^0(S_K^{tor}(v), \omega^{w(z),\dagger}(-D))$, with $K = K^pI$, and I a Iwahori subgroup at p, (which is a space of overconvergent cuspidal forms, defined in [**Her19**] Definition 6.12). Moreover if $\kappa \in \mathcal{W}(\mathbb{C}_p)$ is an algebraic character of M-dominant coherent weight, then the action of $\mathcal{H}^S$ preserves the subspace of classical forms $H^0(S_K^{tor}, \omega^\kappa(-D))$ and coincides with the "usual" action of $\mathcal{H}^S$ on $H^0(S_K^{tor}, \omega^\kappa(-D))$.

We remind now that $\mathcal{E}'$ contains an accumulation and Zariski dense subspace of automorphic points that we will call *very regular small slope classical points.*

Let $\mathcal{Z} \subset \mathcal{E}'(\mathbb{C}_p)$ be the set of points z satisfying [**Her19**] Proposition 8.2, a slightly stronger form of Theorem 8.3, namely

$$\max(n_\tau + v_p(\alpha_\tau), 0) < \inf_\tau(k_{\sigma_\tau,p_\tau} + k_{\overline{\sigma_\tau},q_\tau}), \quad \forall \tau \in \Sigma_F \tag{3}$$

where $w(z)$ is (thus) a G-dominant algebraic character of coherent weight $(k_{\sigma,i})_{\substack{\sigma\in\Sigma_E \\ 1\leqslant i\leqslant p_\sigma}}$ and α_τ is the eigenvalue for the operator $U_{v,\min(p_\tau,q_\tau)}$ and n_τ is a normalisation constant depending only on $(p_\tau, q_\tau)_\tau$, asking that $w(z)$ is moreover *far from the walls* as in [**Har90a**] Lemma 3.6.1. For $C > 0$, we define $\mathcal{Z}_C \subset \mathcal{Z}$ adding the condition $k_{\sigma,i} - k_{\sigma,i+1} > C$ for all i. For $C >> 0$, these points give rise to crystalline representations at p as we will see. Each of the sets $\mathcal{Z}_C$ is accumulation in $\mathcal{Z}$, we can thus prove the claim with $\mathcal{Z}_C$ replaced by $\mathcal{Z}$. By [**Bij16**] (see [**Her19**, Thm 9.4]), if $z \in \mathcal{Z}$, the system of eigenvalues corresponding to z has an eigenvector in $H^0(S_K^{tor}, \omega^\kappa(-D))$. This implies that actually $z \in \mathcal{E}'(\overline{\mathbb{Q}}_p)$. It follows from [**SU02**], or [**Har90a**], that there exists a cuspidal automorphic form π of $G(\mathbb{A}_F)$ such that $\pi_f^{K^p} \neq 0$, the Satake parameter of π_v, $v \nmid p$, corresponds to $\lambda|_{\mathcal{H}_v} \otimes k(z)$ and π_∞ is tempered of weight $((k_{\sigma_\tau,p_\tau},\cdots,k_{\sigma_\tau,1}, -k_{\overline{\sigma_\tau},1},\cdots,-k_{\overline{\sigma_\tau},q_\tau}))_{\tau\in\Sigma_F} - \rho_G - w_{0,M}\rho_G$, with ρ_G the half-sum of positive roots, as we will explain.

To be able to clearly label the weights, let P_h[17] be the parabolic corresponding to $\mathfrak{p}$, and choose a Borel, equivalently a set Φ^+ of positive roots such that if Φ^+_c is the set of positive roots contained in the Levi of $\mathfrak{p}$, then $\Phi^+_{nc} := \Phi^+ \backslash \Phi^+_c$ is chosen to be included in $\mathfrak{g}_\mathbb{C}/\mathfrak{p}$. Equivalently $B \subset P_h^{opp} = P_\mu$, the parabolic opposite to P_h. This allows us to label similarly (classical, dominant) weights of representations of K_∞ (with respect to Φ^+_c) and of G (with respect to Φ^+). Let us be more precise for these choices. Let $G(\mathbb{Q}_p)$ our unitary group, thus given by the hermitian form J, and let T be its diagonal torus, and T^1 the subtorus of elements of similitude 1. We have an embedding

$$\begin{array}{ccc} T^1(\mathbb{Z}_p) & \longrightarrow & \prod_{\tau\in\Phi} \mathrm{GL}_{p_\tau,K} \times \mathrm{GL}_{q_\tau,K} \\ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} & \longmapsto & \left(\begin{pmatrix} \sigma_\tau(a_{p_\tau})^{-1} & & \\ & \ddots & \\ & & \sigma_\tau(a_1)^{-1} \end{pmatrix}, \begin{pmatrix} \overline{\sigma_\tau}(a_{q_\tau})^{-1} & & \\ & \ddots & \\ & & \overline{\sigma_\tau}(a_1)^{-1} \end{pmatrix} \right)_\tau \end{array}$$

[16] There $\mathcal{H}^S$ is denoted $\mathcal{H}_{K^p}$. See also remark 7.12 of [**Her19**]
[17] also called P_μ^{std} in other references

where $a_i \in \mathcal{O}_E \otimes \mathbb{Z}_p$ and $a_i\overline{a_{n+1-i}} = 1$. Writing $a_i = (z_i, t_i)$ (using the choice of $w|v$ for all v and z_i corresponding to w in Φ), we can rewrite the previous embedding

$$\iota : \begin{array}{ccc} ((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times)^n & \longrightarrow & \prod_{\tau \in \Phi} \mathrm{GL}_{p_\tau,K} \times \mathrm{GL}_{q_\tau,K} \\ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} & \longmapsto & \left(\left(\begin{pmatrix} \tau(z_{p_\tau})^{-1} & & \\ & \ddots & \\ & & \tau(z_1)^{-1} \end{pmatrix}, \begin{pmatrix} \tau(z_{p_\tau+1}) & & \\ & \ddots & \\ & & \tau(z_n) \end{pmatrix} \right) \right)_\tau \end{array}$$

We choose the diagonal torus and upper Borel for $M \simeq \prod_\tau \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$ [18], which we see as the Levi for $P_\mu \subset \prod_{\tau\in\Phi} \mathrm{GL}_{p_\tau+q_\tau}$, the standard lower parabolic with blocs p_τ, q_τ. Thus the choice of Φ^+ for G corresponds to the standard upper Borel. Denote ρ_G the half-sum of positive roots in Φ^+.

Thus if we choose $\kappa := (k_{\sigma,i})_{\sigma\in\Sigma_E, 1\leqslant i\leqslant p_\sigma}$ a dominant (integral) weight for M for the previous upper triangular Borel, then the algebraic representation of highest weight κ is constructed using $-w_{0,M}\kappa$, which induces the weight χ of $((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times)^n$ which is algebraic of coherent weight κ in the sense of equation (2). This weight χ is Φ^+-dominant if and only if $k_{\sigma_\tau,p_\tau} \geqslant -k_{\overline{\sigma_\tau},q_\tau}$.

Let $z \in \mathcal{Z}$, which corresponds to a classical automorphic form (itself giving an automorphic representation π) appearing in $H^0(S_K^{tor}, \omega^\kappa)$ for $\kappa = (k_{\sigma,i})_{\sigma\in\Sigma_E,1\leqslant i\leqslant p_\sigma}$ as before, which is a classical (and M-dominant) weight. Then $\chi = w(z) \in \mathcal{W}$ is the algebraic character of weight κ, i.e. is $-w_{0,M}\kappa \circ \iota$. This sheaf ω^κ coincides with the coherent sheaf V_s over $\mathbb{C}$ defined by Harris ([**Har90b**]), associated to the highest weight s representation of $M = \prod_{\tau\in\Phi} \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$, with $s = (-k_{\tau,p_\tau}, \dots, -k_{\tau,1}, k_{\overline{\tau},1}, \dots, k_{\overline{\tau},q_\tau})$ with the previous identifications. This calculation is the one done in [**FP19**] section 7.4, based on [**Gol14**]. Remark that if χ is algebraic of weight κ and dominant, then the dominant representative of $-s$ is given by $\mathrm{wt}(\chi) = (k_{\sigma_\tau,1}, \dots, k_{\sigma_\tau,p_\tau}, -k_{\overline{\sigma_\tau},q_\tau}, \dots, -k_{\overline{\sigma_\tau},1})$. In particular as the Hecke eigensystem corresponding to z appears in $H^0(S_K^{tor}, \omega^\kappa)$ thus in $H^0(S_K^{tor}(\mathbb{C}), V_s)$ this means that

$$H^0(\mathfrak{p}, K_\infty, \pi_\infty \otimes V_s) \neq \{0\},$$

i.e. that the infinitesimal character of π_∞ is $-s - \rho_G$ (up to reordering) by e.g. [**Har90b**] Proposition 4.3.2 (see also [**BP20**] Proposition 5.37). But if $z \in \mathcal{Z}$ and π an automorphic representation corresponding to its system of eigenvalues $\lambda(z)$ of $\mathcal{H}^S$, then using that $w(z)$ is far from the walls, by Corollary A.8 there is a semisimple representation $\rho^u = \rho^u_z : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ such that $\rho^u(\mathrm{Frob}_v)$ is associated to the semi-simple conjugacy class at v determined by λ, for all $v \notin S$ and satisfying moreover

$$(\rho^u)^\vee \simeq (\rho^u)^c \otimes \varepsilon^{n-1}.$$

By for example [**BGGT14**], the previous calculation of the infinitesimal weight means that ρ^u, associated to π, has Hodge-Tate weights given by $-s-\rho_G-\frac{n-1}{2}$ i.e. the v, τ Hodge-Tate weights of ρ^u_z are (up to order)

$$(k_{v,\tau,p_\tau} + 1 - n, k_{v,\tau,p_\tau-1} + 2 - n, \dots, k_{v,\tau,1} + p_\tau - n, -k_{v,\overline{\tau},1} + p_\tau - n + 1, \dots, -k_{v,\overline{\tau},q_\tau})$$

[18] This is actually the subgroup of M of element with similitude factor 1, but in all this discussion we ignore the similitude factor to simplify the notations.

which we can reorder to be dominant for very regular $z \in \mathcal{Z}$,

(4)
$$(k_{v,\tau,1}+(p_\tau-n), k_{v,\tau,2}+(p_\tau-n-1), \dots, k_{v,\tau,p_\tau}+1-n, -k_{v,\overline{\tau},q_\tau}, \dots, -k_{v,\overline{\tau},1}+(p_\tau-n+1)).$$

We first construct a union of connected components of $\mathcal{E}'$ and a map from this subspace to $\mathcal{X}_{\overline{\rho}}^{pol}$. As in **[Che04]**, we construct a determinant

$$D : G_E \longrightarrow \mathcal{O}_{\mathcal{E}'}.$$

Let $z \in \mathcal{Z}$ and π an automorphic representation corresponding to its system of eigenvalues $\lambda(z)$ of $\mathcal{H}^S$, as we have seen, by Corollary A.8 there is a semisimple representation $\rho^u : G_E \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ associated to $\lambda(z)$ such that

$$(\rho^u)^\vee \simeq (\rho^u)^c \otimes \varepsilon^{n-1}.$$

Let D_z the pseudo-representation of ρ^u_z. The continuous map $(D_z)_{z\in\mathcal{Z}}$ from G_E to $\prod_{x\in\mathcal{Z}} k(x)$ factors actually through $\Gamma(\mathcal{E}', \mathcal{O}^+_{\mathcal{E}'})$ and gives rise to a pseudo deformation D on $\Gamma(\mathcal{E}', \mathcal{O}^+_{\mathcal{E}'})$. By continuity, we have $D^\vee \simeq D^c\varepsilon^{n-1}$.

As there is only a finite number of possible reductions modulo p of D, there is $\mathcal{E}'(\overline{\rho})$ an open and closed subset of $\mathcal{E}'$ of points whose reduction of D is $(\mathrm{tr})\overline{\rho}$. This is non empty by Hypothesis 4.2. In particular the restriction of the previous D to $\mathcal{E}'(\overline{\rho})$ induces a morphism of rigid analytic spaces

$$\mathcal{E}'(\overline{\rho}) \longrightarrow \mathcal{X}^{pol}_{\overline{\rho}}.$$

Now we construct a rigid analytic map $\mathcal{E}' \to \mathcal{T}$.

Denote $w = (w_{v,1}, \dots, w_{v,n})_v$ the universal character of $((\mathcal{O}_F \otimes \mathbb{Z}_p)^\times)^n$. In **[Her19**, Section 7.2.2] we constructed Hecke operators which are in $\mathcal{O}(\mathcal{E}')^\times$, denoted by $U_{v,i}$ for $v|p$ in F and $i = 0, \dots, n$. The operator $U_{v,i}$ coincides up to normalisation (this normalisation is made in order to vary in family) with the double class,

$$\left(\begin{pmatrix} I_i & \\ & p^{-\delta_{v=v'}} I_{n-i}\end{pmatrix}_w, \begin{pmatrix} I_{n-i} & \\ & p^{-\delta_{v=v'}} I_i\end{pmatrix}_{\overline{w}}, \right)_{w\overline{w}=v'|\ p} \in G(\mathbb{Q}_p) \subset \prod_{w\overline{w}=v'|p \text{ in F}} \mathrm{GL}_n(E_w)\times\mathrm{GL}_n(E_{\overline{w}}).$$

If $U^{class}_{v,i}$ denotes the action of the (classical, i.e. non normalised) Hecke operator corresponding to the previous Iwahori double class acting on global sections of the classical automorphic sheaf, for (fixed) algebraic weight $w \in \mathcal{W}$, then the normalisation is

$$U_{v,i} = \widetilde{w}_v \begin{pmatrix} 1 &&&&& \\ & \ddots &&&& \\ && 1 &&& \\ &&& p^{-1} && \\ &&&& \ddots & \\ &&&&& p^{-1}\end{pmatrix} U^{class}_{v,i},$$

where $\widetilde{w}_v$ is the (unique) algebraic extension of w_v as a character of $(F_v^\times)^{n}$ (19), and there is i times 1 (and $n-i$ times p^{-1}) appearing in the matrix (see before and remark 7.5,

(19) This normalisation factor, which is a power of p, comes when trying to express a character of T, the maximal torus of G, as one of $F \otimes \mathbb{Q}_p^\times$. It would be simpler here to explain this normalisation using T. The key point is

together with remark 8.3 of [**Her19**]). For all $i \in \{1, \dots, n\}$, we set

$$F_{v,i} := U_{v,i} U_{v,i-1}^{-1}.$$

It corresponds, up to normalisation, to the Hecke operator in $\mathcal{A}(p)$[20]

$$F_{v,i}^{cl} := \left(\begin{pmatrix} I_{i-1} & & \\ & p^{\delta_{v=v'}} & \\ & & I_{n-i} \end{pmatrix}_w , \begin{pmatrix} I_{n-i} & & \\ & p^{-\delta_{v=v'}} & \\ & & I_{i-1} \end{pmatrix}_{\overline{w}} , \right)_{w\overline{w}=v' \mid p} \in G(\mathbb{Q}_p),$$

and the normalisation is the following, for w algebraic of infinitesimal weight $h = (h_{\tau,i})_{\tau,i}$

$$F_{v,i} = \widetilde{w}_v \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & p & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} F_{v,i}^{cl} = p^{\sum_\tau h_{i,\tau}} F_{v,i}^{cl},$$

where p is in position i. The all point is that $F_{v,i}^{cl} \notin \mathcal{O}(\mathcal{E}')$, i.e. they don't interpolate, whereas $F_{v,i} \in \mathcal{O}(\mathcal{E}')^\times$. We construct characters $\delta_{v,i}^0 : F_v^\times \longrightarrow \mathcal{O}(\mathcal{E}')^\times$ by setting $\delta_{v,i}^0(p) := F_{v,i}$ and

$$(\delta_{v,i}^0)_{|\mathcal{O}_{F_v}^\times} = w_{v,i},$$

and we finally set

$$\delta_{v,i} := \delta_{v,i}^0 \times \prod_\tau x_\tau^{s_\tau(i)} \times |.|^{\frac{1-n}{2}}, \tag{5}$$

where for $\tau : F_v \hookrightarrow \mathbb{C}_p$, $x_{\tau|\mathcal{O}_{F_v}} = \tau_{\mathcal{O}_{F_v}}$ and $x_\tau(p) = 1$, and $s_\tau(i) = \frac{1-n}{2} + p_\tau - i$ if $1 \leqslant i \leqslant p_\tau$, and $s_\tau(i) = \frac{n-1}{2} - (i - p_\tau - 1)$ if $i > p_\tau$[21]. Thus the characters $(\delta_{v,i})_{v,i}$ gives a map

$$\mathcal{E}' \longrightarrow \mathcal{T}.$$

Still denote $\mathcal{Z}$ for $\mathcal{Z} \cap \mathcal{E}'(\overline{\rho})$, which is Zariski dense and accumulation. Now we are reduced to prove that the two constructed maps $\mathcal{E}'(\overline{\rho}) \longrightarrow \mathcal{X}_{\overline{\rho}}^{pol}$ and $\mathcal{E}'(\overline{\rho}) \longrightarrow \mathcal{T}$ are compatible, in the sense that for $z \in \mathcal{Z}$ the second map is the parameter of a triangulation for the image of z via the first map. By local global compatibility at v for π and ρ^u, we have that, using $\pi_v^I \neq \{0\}$ by construction of $\mathcal{E}'$, that π_v is a subquotient of the Borel induction of an unramified character χ of $(F_v^\times)^n$ (e.g. [**BC09**] Proposition 6.4.3 and 6.4.4) with χ related to the eigenvalues of $F_{v,i}^{class}$ at z ($\chi = (\varphi_1, \dots, \varphi_n) = (F_{v,1}^{cl}, \dots, F_{v,i}^{cl})\delta_B^{1/2}$). But $F_{v,i}$ has a locally constant valuation (thus not $F_{v,i}^{cl}$!), so up to choose another point of $\mathcal{Z}$ close to z, we can assume that this induced representation is irreducible, and thus

that the double class corresponding to $U_{v,i}$ has similitude factor p^{-1}, thus is not in T^1. But remark that then the Hecke operator $F_{v,i}$ has similitude factor 1.

[20] But not the the double class associated to the following matrix, as the Hecke algebra at Iwahori level is not commutative! See e.g. [**BC09**] Proposition 6.4.1 and the remark that follows.

[21] Remark that actually in our quasi split situation we have p_τ, q_τ which doesn't depend on $\tau \in \Phi$. In any case $(s_\tau(1), \dots, s_\tau(n))_\tau = w_{0,M}(0, \dots, n-1)_\tau + \underline{\frac{1-n}{2}} = -w_{0,M}\rho_G$, with ρ_G defined in the next paragraph.

unramified. By local global compatibility this proves that there is an accumulation subset of $\mathcal{Z}$, which accumulates at any point of $\mathcal{E}$ with algebraic weight, consisting of points z with representation ρ_z semi simple corresponding to D_z, crystalline at every $v|p$ and such that $D_{cris}(\rho_v)$ has all its refinement, one of which is given by $(D_{cris}(\delta_{v,i}))_{1\leqslant i\leqslant n}$, i.e. $(F^{cl}_{v,i})_i$. Moreover, the calculation for $z \in \mathcal{Z}$ we did in equation 4, together with the definition of the weight of δ in equation 5 implies that the Hodge-Tate weights of ρ_z are given by $\delta_{|(\mathcal{O}_F\otimes\mathbb{Z}_p)^\times}$, in the right order ! This means that the map

$$D \times \delta : \mathcal{E}(\overline{\rho}) \longrightarrow \mathcal{X}^{pol}_{\overline{\rho}} \times \mathcal{T},$$

sends a dense subset of $\mathcal{Z}$ (namely the previous one where points are crystalline) into $\mathcal{Z}'_{K^p}$, but conversely by construction of $\mathcal{E}'$, all points of $\mathcal{Z}'_{K^p}$ are in the image of the previous map.

Moreover $D\times\delta$ is a closed immersion. Indeed, by construction $\mathcal{E}'$ is (locally) constructed as the image of $\mathcal{H}\otimes\mathcal{O}_{\mathcal{W}} = \mathcal{H}^S\otimes\mathcal{O}_{\mathcal{T}}$ where $\mathcal{H} = \mathcal{H}^S\otimes\mathcal{A}(p)$ acting on some space of overconvergent, locally analytic modular forms (of finite slope). Let $U \subset \mathcal{X}^{\chi-pol}_{\overline{\rho}} \times \mathcal{T}$ be an affinoid, in particular it is quasi-compact thus the slopes of $\mathcal{A}(p)$ on U are bounded (say by α). Thus $(D\times\delta^{-1})(U)$ is included in $\mathcal{E}^{\leqslant\alpha}_{v,w}$ for some v, w (see [**Her19**]) and then, by local-global compatibility, it is clear that $(D\times\delta)^{-1}(U) \longrightarrow U$ is a closed immersion. As explained, $\mathcal{Z}$ accumulates to any point with classical weight of $\mathcal{E}'$, thus to $\mathcal{Z}'$. If we denote by h the composite of the map

$$\mathcal{E} \longrightarrow \mathcal{T} \longrightarrow \mathcal{W},$$

it coincides with a map

$$\mathcal{E} \xrightarrow{w} \mathcal{W} \xrightarrow{\phi} \mathcal{W},$$

where ϕ is the isomorphism of $\mathcal{W}$ given by the definition (5). The properties of the map h thus comes from the analogous one for w, proven in [**Her19**] Theorem 9.5 (see also [**Che04**] which was the first to prove those properties). ☐

From now on, to lighten notations denote $\mathcal{E}(\overline{\rho}) := \mathcal{E}_{K^p}(\overline{\rho})$, $\mathcal{F}(\overline{\rho}) := \mathcal{F}_{K^p}(\overline{\rho})$ $\mathcal{Z}' := \mathcal{Z}'_{K^p}$ accordingly(22).

5. Automorphic forms, infinite fern and big image

Let K be finite extension of $\mathbb{Q}_p$ and $\overline{K}$ an algebraic closure of K. Denote v_p the p-adic absolute value of $\overline{K}$ such that $v_p(p) = 1$. Let $K_0 \subset K$ be the maximal unramified extension of $\mathbb{Q}_p$ with Frobenius operator σ and set $f := [K_0 : \mathbb{Q}_p]$. Let L be a finite extension of $\mathbb{Q}_p$ such that $K\otimes_{\mathbb{Q}_p} L \simeq L^{[K:\mathbb{Q}_p]}$. If (ρ, V) is a crystalline representation of $\mathrm{Gal}(\overline{K}/K)$, we denote $(D_{\mathrm{cris}}(V), \varphi)$ its associated φ-module. It is a finite dimensional free $K_0\otimes_{\mathbb{Q}_p} L$-module of rank $n = \dim_L V$ with a $\sigma\otimes\mathrm{Id}_L$-linear automorphism φ. Its de Rham module $D_{\mathrm{dR}}(V)$ is a filtered finite free $K\otimes_{\mathbb{Q}_p} L$-module. The ***Hodge-Tate type*** of V is the $[K:\mathbb{Q}_p]$-uple $(k_{1,\tau} \geqslant \cdots \geqslant k_{n,\tau})_{\tau:K\hookrightarrow L}$ where the $k_{i,\tau}$ are the integers m such that $\mathrm{gr}^{-m}(D_{\mathrm{dR}}(V)\otimes_{K,\tau} L) \neq 0$ counted with multiplicity.

(22) Everything we will say is still dependant of this level K^p. At the end of the article, in corollary 8.11, we show that there exists an optimal level K^p, but we can't choose it right away. Compare [**Che11**] Lemma 2.4

Definition 5.1. — We say that a crystalline Galois representation (ρ, V) over L is ***Hodge-Tate regular*** (or simply ***HT-regular***) if, for all $\tau : K \hookrightarrow L$, the integers $k_{i,\tau}$ are pairwise distinct. It is said to be ***φ-generic*** if the ***linear*** endomorphism φ^f of the finite free $K_0 \otimes_{\mathbb{Q}_p} L$-module $D_{\mathrm{cris}}(V)$ is split semisimple regular ie has $\dim_L V$ pairwise distinct eigenvalues φ_i in L such that $\varphi_i/\varphi_j \notin \{1, p^f\}$ (note that these eigenvalues are in L since φ^f commutes to the semilinear action of $\mathrm{Gal}(K_0/\mathbb{Q}_p)$ on $D_{\mathrm{cris}}(V)$ given by φ).

Remark 5.2. — In [**Che11**, §3] Chenevier introduced the notion of *weakly generic* crystalline (φ, Γ)-module, i.e. crystalline (φ, Γ)-modules for which all refinements are non-critical, and *weakly-generic* crystalline (φ, Γ)-module for which n (the rank of the (φ, Γ)-module) well-positioned refinements are non critical. It is possible to deduce from the results of [**Che11**] that if the classical points of $\mathcal{E}$ or $\mathcal{X}_{\overline{\rho}}^{pol}$ which are weakly-generic at p are Zariski-dense, then modular points are dense in $\mathcal{X}_{\overline{\rho}}^{pol}$ (or the closure has at least the expected dimension). He moreover proves that if $n = 3$, every φ-generic, HT regular absolutely irreducible (φ, Γ) is automatically weakly generic. Unfortunately it does not seem to be true anymore even for $n = 4$ that absolutely irreducible points are weakly generic, as shown in the following example, and thus it does not seem any easier to prove that weakly generic points are dense in $\mathcal{X}_{\overline{\rho}}^{pol}$ when $n \geqslant 4$ than proving the analogous result for generic points.

Example 5.3. — Let $(V, \varphi, \mathrm{Fil}^\bullet)$ be the filtered φ-module of an irreducible, φ-generic, HT regular, crystalline 4 dimensional representation of $G_{\mathbb{Q}_p}$ with L-coefficients. Choosing L big enough, we can assume that there exists a basis (f_1, f_2, f_3, f_4) of V such that $\varphi(f_i) = \varphi_i f_i$. Refinements of V are thus given by a permutation σ of this basis. By irreducibility and weak-admissibility, we check that it is impossible for $\mathrm{Fil}^k V \neq \{0\}, V$ to be φ-stable. For example, suppose that the HT weights are $-k_4 < -k_3 < -k_2 < -k_1$ (i.e. the jumps of the filtration on V are $k_1 < k_2 < k_3 < k_4$) with

- $\mathrm{Fil}^{k_1} V =< f_2, f_3, f_1 + f_4 >=< f_2, f_2 + f_3, f_1 + f_4 >$
- $\mathrm{Fil}^{k_2} V =< f_1 + f_4, f_2 + f_3 >$
- $\mathrm{Fil}^{k_3} V =< f_1 + f_4 >$
- $\mathrm{Fil}^{k_4} V = \{0\}$.

We can check that the non critical refinements are given by $\sigma = \mathrm{id}, (23), (14), (14)(23)$, and they don't form a weakly-nested sequence. Moreover, for generic choices of k_i and $v(\varphi_i)$, the associated p-adic representation $V(D)$ will be irreducible. Indeed, if $v(\varphi_1) = v(\varphi_2) = v(\varphi_3) = 16, v(\varphi_4) = 12, k_1 = 0, k_2 = 10, k_3 = 20, k_4 = 30$, then we can check that no non-trivial φ-stable submodule of D is weakly-admissible. In particular if we do not already know that generic points are Zariski dense, it is not likely to prove that weakly generic ones are.

Moreover we can check that locally the image of the tangeant space of those refined points doesn't cover all the tangent space for the corresponding point in the local deformation ring (i.e. at this point the analogous of proposition 7.5 only for non-critical refinements isn't true). In the following we will use a replacement of those generic points by the so-called *almost generic* ones, which are Zariski dense in the fern and for which we can apply proposition 7.5 and thus Chenevier's stategy. Remark that some irreducible weakly-admissible filtered φ-modules of dimension 4 which admits a critical refinement are weakly-generic, but we don't know how to discriminate these from the previous example on the deformation rings (or on the infinite fern).

Denote by $\overline{\mathcal{F}(\overline{\rho})}$ the Zariski-closure of the image $\mathcal{E}(\overline{\rho}) \to \mathfrak{X}_{\overline{\rho}}^{\chi-pol}$, i.e. the Zariski closure of the infinite fern $\mathcal{F}(\overline{\rho})$.

Definition 5.4. — We say that a Galois representation $\rho : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ has enormous image, if $\rho(G_{E(\zeta_{p^\infty})})$ is enormous, in the sense of [**NT19**] Definition 2.27. We say that a point x of $\mathcal{E}(\overline{\rho})$ (resp. of $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$) is ***almost generic*** if it is in $\mathcal{Z}$ (resp. in the image of $\mathcal{Z}$ in $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$), the associated Galois representation ρ has enormous image, and if $\rho_{|G_{E_v}}$ is crystalline, φ-generic and HT-regular for all $v|p$.

Let v be a place of E dividing p. Let (ρ, V) be a continuous finite dimensional representation of G_{E_v} over L. It follows from the compactness of G_{E_v} that $\rho(G_{E_v})$ is a closed subgroup of $\mathrm{GL}_L(V)$. The p-adic analogue of Cartan's Theorem shows that $\rho(G_{E_v})$ is a p-adic Lie subgroup of $\mathrm{GL}_L(V)$ so that we can define $\mathfrak{g}_\rho := \mathrm{Lie}(\rho(G_{E_v}))$ (see [**Ser89**, Rem. I.1.1]) and $\mathfrak{g}_{\rho,L}$ the L-span of $\mathfrak{g}_\rho$ in $\mathrm{End}_L(V)$. Our goal is to prove that almost generic points are Zariski-dense in the infinite fern $\mathcal{F}(\overline{\rho})$. Such a result was proven by Taïbi ([**Tai16**]) in a slightly different (and more difficult) context (improving results of [**BC09**]). As the case we consider is easier, and for convenience of the reader, we repeat the argument in our context.

Denote by K_n/E_v the compositum of extensions of degree dividing n, this is a finite Galois extension.

Proposition 5.5. — *Let (ρ, V) be some continuous n-dimensional representation of G_{E_v} over L and assume that $(\rho|_{G_{K_n}}, V)$ is absolutely irreducible. Then V is a simple $\mathfrak{g}_{\rho,L}$-module, $\mathfrak{g}_{\rho,L}$ is a reductive Lie algebra and $\mathfrak{h}_{\rho,L}$, the semisimple part of $\mathfrak{g}_{\rho,L}$, is isomorphic to a sub Lie algebra of $\mathfrak{sl}(V)$ of semisimple rank at most $\dim_L V - 1$.*

Proof. — Let K be a finite extension of E_v, we claim that $\rho|_{G_K}$ is absolutely irreducible. We can assume that K/E_v is Galois. Suppose that $\rho|_{G_K}$ is not absolutely irreducible, then as

$$\rho : G_{E_v} \longrightarrow \mathrm{GL}_L(V) \simeq \mathrm{GL}_n(L)$$

is absolutely irreducible and G_K is normal in G_{E_v}, we have, up to enlarge L, a decomposition into absolutely irreducible G_K-representations

$$\rho|_{G_K} = \bigoplus_{k=1}^{r} W_k,$$

and G_{E_v} permutes these representations, in particular they have the same dimension. Let $H \subset G_{E_v}$ be the stabiliser of W_1, then G_{E_v}/H acts transitively on the W_k and thus $[G_{E_v} : H]\dim(W_1) = n$. In particular $H = G_{K'}$ with K' a finite extension of E_v of degree dividing n, thus $K' \subset K_n$, but $\rho|_{G_{K_n}}$ is irreducible, thus as is $\rho|_{G_{K'}}$ and thus $r = 1$ i.e. $\rho|_{G_K}$ is irreducible.

For each open subgroup $H \subset G_{E_v}$, the representation $(\rho|_H, V)$ is irreducible, so that V is a simple $\mathfrak{g}_{\rho,L}$-module. Thus $\mathfrak{g}_{\rho,L}$ has a simple faithful module, which implies that

$\mathfrak{g}_{\rho,L}$ is a reductive Lie algebra. Let $\mathfrak{g}_{\rho,L} = \mathfrak{a}_{\rho,L} \oplus \mathfrak{h}_{\rho,L}$ be the decomposition of $\mathfrak{g}_{\rho,L}$ as direct sum of an abelian and of a semisimple Lie algebras. As V is an absolutely simple $\mathfrak{g}_{\rho,L}$-module, the Lie algebra $\mathfrak{a}_{\rho,L}$ acts on V by scalars and V is an absolutely simple $\mathfrak{h}_{\rho,L}$-module. Then $\mathfrak{a}_{\rho,L} \subset L\,\mathrm{Id}_V$ and, as $\mathfrak{h}_{\rho,L}$ is semisimple, $\mathfrak{h}_{\rho,L} \subset \mathfrak{g}_{\rho,L} \cap \mathfrak{sl}(V)$. As a consequence $\mathfrak{h}_{\rho,L} = \mathfrak{g}_{\rho,L} \cap \mathfrak{sl}(V)$ is a semisimple Lie algebra and V is an absolutely simple $\mathfrak{h}_{\rho,L}$-module. As $\mathfrak{sl}(V)$ has rank $\dim_L V - 1$, the rank of $\mathfrak{h}_{\rho,L}$ is at most $\dim_L V - 1$. $\square$

Proposition 5.6. — *There exist finitely many nonzero $\mathbb{Q}$-linear forms $\Lambda_1, \dots, \Lambda_r$ on $\mathbb{Q}^n$ such that the following is true: let (ρ, V) be a crystalline n-dimensional representation of G_{F_v} over L, with Hodge-Tate weights $(k_{1,\sigma} \leqslant \cdots \leqslant k_{n,\sigma})_\sigma$ such that $(\rho|_{G_{K_n}}, V)$ is absolutely irreducible and such that there exists at least one $\sigma : F_v \hookrightarrow L$ such that for all $1 \leqslant i \leqslant r$, $\Lambda_i(k_{1,\sigma}, k_{2,\sigma}, \dots, k_{n,\sigma}) \neq 0$, then $\rho(G_{F_v})$ contains an open subgroup of $\mathrm{SL}(V)$.*

Proof. — Let C be some algebraically closed field of characteristic 0. The classification of semisimple Lie algebras and their representations shows that all semisimple Lie algebras and their finite dimensional simple modules are defined over $\mathbb{Q}$, that there are finitely many isomorphism classes of semisimple Lie algebras of bounded rank and that each of them has finitely many semisimple modules of bounded rank. Consequently, for a fixed $n \geqslant 2$, there exist a finite number of pairs $(\mathfrak{h}_i, \theta_i)$ where $\mathfrak{h}_i$ is a semisimple Lie algebra and θ_i an embedding of $\mathfrak{h}_i$ in $\mathfrak{sl}_{n,\mathbb{Q}}$ such that for each semisimple Lie subalgebra $\mathfrak{h} \subset \mathfrak{sl}_{n,C}$, there exists i such that $\mathfrak{h} \simeq \mathfrak{h}_i \otimes_{\mathbb{Q}} C$ and the inclusion is $\mathrm{GL}_n(C)$-conjugated to $\theta_i \otimes \mathrm{Id}_C$. As a consequence a Cartan subalgebra of $\mathfrak{h}$ is conjugated to one of finitely many $\mathbb{Q}$-linear subspaces of the space of diagonal matrices in $\mathfrak{sl}_{n,C}$. Moreover it follows from [**Bou**, VIII.§3 Prop.2.(ii)] and from Borel-de Siebenthal Theorem ([**Kan01**, Thm. 12.1]) that a semisimple Lie subalgebra of $\mathfrak{sl}_{n,C}$ containing $\mathfrak{h}$ is equal to $\mathfrak{sl}_{n,C}$ or of rank strictly less than $n-1$. Thus there exist finitely many nonzero $\mathbb{Q}$-linear forms $\Lambda'_1, \dots, \Lambda'_s$ on $\mathbb{Q}^n$ such that if $\mathfrak{h}$ a semisimple subalgebra of $\mathfrak{sl}_{n,C}$ of rank strictly less than $n-1$ and $x \in \mathfrak{h}$ is a semisimple element of eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicities), then there exists $1 \leqslant i \leqslant s$ and $w \in \mathfrak{S}_n$ such that $w(\Lambda'_i)(\lambda_1, \dots, \lambda_n) := \Lambda'_i(\lambda_{w(1)}, \dots, \lambda_{w(n)}) = 0$. We set

$$\{\Lambda_1, \dots, \Lambda_r\} = \{w(\Lambda'_i) \mid 1 \leqslant i \leqslant s,\ w \in \mathfrak{S}_n\}.$$

Let $\Theta \in \mathrm{End}_{\mathbb{C}_p \otimes_{\mathbb{Q}_p} L}(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)$ the Sen operator of V. As (ρ, V) is Hodge-Tate, it follows from [**Sen73**, Thm. 1] that Θ belongs to $\mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathfrak{g}_{\rho,L} \subset \mathbb{C}_p \otimes_{\mathbb{Q}_p} \mathrm{End}_{\mathbb{Q}_p} V$.

Suppose L is big enough so that $F_v \otimes_{\mathbb{Q}_p} L \simeq L^{[F_v:\mathbb{Q}_p]}$. Then

$$\mathbb{C}_p \otimes_{\mathbb{Q}_p} L = \prod_{\sigma : F_v \hookrightarrow L} \mathbb{C}_p \otimes_{F_v,\sigma} L,$$

decomposes over all embeddings of F_v and let Θ_σ be the $\mathbb{C}_p \otimes_{F_v,\sigma} L$-linear endomorphism of $\mathbb{C}_p \otimes_{F_v,\sigma} V$ induced by Θ. The eigenvalues of Θ_σ are the σ-Hodge-Tate weights $(k_{1,\sigma} \leqslant \cdots \leqslant k_{n,\sigma})$ of (ρ, V) (counted by multiplicities).

Assume that $\Lambda_i(k_{1,\sigma}, \dots, k_{n,\sigma}) \neq 0$ for all $1 \leqslant i \leqslant r$. Then by what preceeds, the element Θ_σ can't be contained in a strict semisimple Lie subalgebra of $\mathbb{C}_p \otimes_{F_v,\sigma} \mathfrak{sl}(V)$ so that $\mathbb{C}_p \otimes_{F_v,\sigma} \mathfrak{h}_{\rho,L} = \mathbb{C}_p \otimes_{F_v,\sigma} \mathfrak{sl}(V)$. For dimension reasons, we have $\mathfrak{h}_{\rho,L} = \mathfrak{sl}(V)$. We conclude that $\rho(G_{F_v})$ contains an open subgroup of $\mathrm{SL}(V)$. $\square$

Proposition 5.7. — *The set of points* $x \in \overline{\mathcal{F}(\overline{\rho})}$ *such that* $\operatorname{Tr}\rho_x|_{G_{K_n}}$ *is absolutely irreducible is a Zariski-dense and Zariski-open subset of* $\overline{\mathcal{F}(\overline{\rho})}$.

Proof. — The fact that the absolutely irreducible locus is Zariski-open is a consequence of [**Che14**, §4.2]. In order to prove that it is Zariski-dense, it is then sufficient to prove that each Zariski-open subset U of $\mathcal{F}(\overline{\rho})$ contains a point x such that $\operatorname{Tr}\rho_x|_{G_{K_n}}$ is absolutely irreducible.

Now we follow the strategy of [**BC11**] and [**Tai16**]. Let us fix some notation. If $x = (\rho_x, \delta_x) \in \mathcal{Z}' \subset \mathcal{E}(\overline{\rho})$ and $\sigma : K_n \hookrightarrow \overline{\mathbb{Q}_p}$, we let $(k_{\sigma,1}(x) \leqslant \cdots \leqslant k_{\sigma,n}(x))$ the Hodge-Tate weights at σ of $\rho_x|_{G_{K_n}}$ and $(\phi_1(x), \cdots, \phi_n(x)) \in k(x)^n$ the ordered eigenvalues of the linearized Frobenius of $D_{\mathrm{cris}}(\rho_x|_{G_{K_n}})$ corresponding to the refinement of $\rho_x|_{G_{F_v}}$ defined by $\delta_{x,v}$. We also set $k_i(x) = \sum_\sigma k_{\sigma,i}(x)$. Let e be the ramification index of $K_n/\mathbb{Q}_p$. The functions $x \mapsto v_p(\phi_i(x)) + e^{-1}k_i(x)$ are therefore locally constant on $\mathcal{Z}'$. Now fix U a Zariski open non empty subset of $\overline{\mathcal{F}(\overline{\rho})}$. We have $U \cap \mathcal{F}(\overline{\rho}) \neq \varnothing$ so that the inverse image V of U in $\mathcal{E}(\overline{\rho})$ is a non empty Zariski-open subset. Let $x \in V \cap \mathcal{Z}'$. Let $c \geqslant \max_i |v_p(\phi_i(x)) + e^{-1}k_i(x)|$ and larger than n^2+1 and let $\mathcal{Z}''_c$ be the subset of point $z \in \mathcal{Z}'$ such that $k_{i+1}(z) - k_i(z) > cen(k_i(z) - k_{i-1}(z))$ $2 \leqslant i \leqslant n-1$, $2 \leqslant i \leqslant n-1$, and $k_2(z) - k_1(z) > 3cen$. If $z \in \mathcal{Z}''_c$, then $|\sum_{i\in I} k_i(z) - \sum_{i\in J} k_i(z)| > cen$ for all distinct non empty proper subsets I and J of the same cardinal in $\{1, \dots, n\}$ (by the same proof than [**BC09**, Lem. 4.5.5]). Then $\mathcal{Z}''_c$ is Zariski dense and accumulates at $\mathcal{Z}'$ in $\mathcal{E}(\overline{\rho})$. Therefore there exists a point $y \in \mathcal{Z}''_c \cap V$ such that moreover $\max_i |v_p(\phi_i(y)) + e^{-1}k_i(y)| \leqslant c$. Then, for $i \neq j$,

$$\begin{aligned}|v_p(\phi_i(y)) - v_p(\phi_j(y))| \geqslant \frac{1}{e}|k_i(y) - k_j(y)| - |v_p(\phi_i(y) + \frac{k_i(y)}{e}| - |v_p(\phi_j(y)) + \frac{k_j(y)}{e}| \\ > 3cn - c - c > 1.\end{aligned}$$

In particular, $\phi_i(y)/\phi_j(y) \neq p$, so if Π is an automorphic representation corresponding to y, we have that Π_v is an irreducible principal series. In particular, all its refinements are accessible. Now we choose a transitive permutation σ of $\{1, \dots, n\}$ and, since all the refinements of Π_v are accessible, there exists $z_0 \in \mathcal{Z}'$ such that $\rho_y \simeq \rho_{z_0}$ and $(\phi_1(z_0), \dots, \phi_n(z_0)) = (\phi_{\sigma(1)}(y), \dots, \phi_{\sigma(n)}(y))$. As in the proof of [**BC11**, Lem. 3.3], we deduce that $\sum_{i\in I} v_p(\phi_i(z_0)) + e^{-1}k_i(z_0)) \neq 0$ for any non empty proper subset I of $\{1, \dots, n\}$. Let $C = \max_i \big|v_p(\phi_i(z_0)) + e^{-1}k_i(z_0)\big|$. Let $\mathcal{Z}'_C$ be the subset of point $z \in \mathcal{Z}'$ such that $k_{\sigma,j+1}(z) - k_{\sigma,j}(z) > C$ for any $\sigma \in \operatorname{Hom}(K_n, \overline{\mathbb{Q}_p})$ and $1 \leqslant i \leqslant n-1$. The set $\mathcal{Z}'_C$ is Zariski-dense and accumulates at $\mathcal{Z}'$ in $\mathcal{E}(\overline{\rho})$ so that there exists a point $z \in V \cap \mathcal{Z}'_C$ such that $\sum_{i\in I}(v_p(\phi_i(z)) + e^{-1}k_i(z)) \neq 0$ for any non empty proper subset I of $\{1, \dots, n\}$. By [**Tai16**, Lem. 3.2.3], if $D' \subset D_{\mathrm{cris}}(\rho_z|_{G_{K_n}})$ is a nonzero proper weakly admissible sub-φ-module, then there exists a non empty proper subset $I \subset \{1, \dots, n\}$ such that

$$\sum_{i\in I} v_p(\phi_i(z)) + e^{-1}\sum_{i\in I} k_i(z) = 0$$

which is not possible. Therefore $\rho_z|_{G_{K_n}}$ is absolutely irreducible. □

Theorem 5.8. — *The set of points* x *in* $\overline{\mathcal{F}(\overline{\rho})}$ *which are in the image of the set* $\mathcal{Z}$, *which are crystalline* φ*-generic and HT-regular and such that* $\rho_x(G_{F_v})$ *contains an open subgroup of*

$\mathrm{SL}(V_x)$ *is a Zariski dense accumulation subset. As a consequence, the set* $\mathcal{X}^{mod,ag} = \{x \in \mathcal{F}(\overline{\rho}) | x$ *is almost generic* $\} \subset \overline{\mathcal{F}(\overline{\rho})}$ *is a Zariski dense accumulation subset.*

Proof. — Let $\Lambda_1, \dots, \Lambda_r$ be nonzero $\mathbb{Q}$-linear forms on $\mathbb{Q}^n$ as in Proposition 5.6. Let σ be some fixed embedding of F_v into K. The set of classical points $x \in \mathcal{E}(\overline{\rho})$ which are crystalline and such that the σ-Hodge-Tate weights of the representation ρ_x are not zeros of all the Λ_i form a Zariski dense accumulation subset in $\mathcal{E}(\overline{\rho})$ (this is a direct consequence of the open image of Proposition 4.6 ; see [**Tai16**] Proposition 2.2.6). As a consequence $\overline{\mathcal{F}(\overline{\rho})}$ is the Zariski-closure of the images of these points in $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$. By Proposition 5.7, the subspace of $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ where $\rho_x|_{G_{K_n}}$ is absolutely irreducible is Zariski-open and Zariski-dense in $\overline{\mathcal{F}(\overline{\rho})}$. We conclude from Proposition 5.6 that the set of classical points x such that ρ_x has an open image is Zariski-dense and an accumulation subset in $\overline{\mathcal{F}(\overline{\rho})}$.

It is enough to prove that classical points such that $\rho_x(G_{F_v})$ contains an open subgroup of $\mathrm{SL}(V_x)$ have enormous image. At such a point x, the Zariski closure of $\rho_x(G_E)$ contains $\mathrm{SL}(V_x)$. As $E(\zeta_{p^\infty})/E$ is abelian, the derived subgroup $\rho_x(G_E)$ is included in $\rho_x(G_{E(\zeta_{p^\infty})})$. By [**Bor91**, I §2.1 (e)], the Zariski closure of $\rho_x(G_{E(\zeta_{p^\infty})})$ contain the derived subgroup of the Zariski closure of $\rho_x(G_E)$ and then contains $\mathrm{SL}(V_x)$. It follows from [**NT19**] Lemma 2.33, that $\rho_x(G_{E(\zeta_{p^\infty})})$ is enormous. □

6. A lemma on Borel enveloppes

This section is independant of the rest of the text, thus its notations should be considered unrelated to the rest also. Fix n an integer, k a field, $G = \mathrm{GL}_{n/k}$, B the upper triangular Borel, T its diagonal torus, $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$ their respective Lie algebras.

Lemma 6.1. — *For every Borel algebra* $\mathfrak{b}'$*, we have*

$$\mathfrak{b}' = \sum_{w \in \mathfrak{C}'_n} (\mathfrak{b}' \cap \mathfrak{b}_w)^{gr=w_0},$$

where $w_0 \in W \simeq \Sigma_n$ *is the longuest element for the order given by* $\mathfrak{b}$*,* $\mathfrak{b}_g = g^{-1}\mathfrak{b}g$*,* $\mathfrak{C}'_n$ *is, up to translation by an element of* $\mathfrak{S}_n$ *depending on* $\mathfrak{b}'$*, the set of "full" cycles :*

$$\mathfrak{C}_n = \{c_{i,j} := (i, i-1, \dots, j+1, j) \in \Sigma_n | i \geqslant j\},$$

and for $g, h \in G$*,*

$$(\mathfrak{b}_g \cap \mathfrak{b}_h)^{gr=w_0} = \{M \in \mathfrak{b}_g \cap \mathfrak{b}_h | hMh^{-1} \pmod{\mathfrak{u}} = \mathrm{Ad}(w_0)(gMg^{-1} \pmod{\mathfrak{u}}) \in \mathfrak{t}\}.$$

Proof. — We can write $\mathfrak{b}' = \mathfrak{b}_g$ for $g \in G$. Let $g = uls$, $u \in \mathfrak{b}, l \in \mathfrak{b}_{w_0}$ lower triangular i.e. $l = w_0 b w_0$ with $b \in B$, and $s \in W$. Thus $\mathfrak{b}_g = (\mathfrak{b}_{w_0})_{bq}$, for $q = w_0 s \in W$. Up to conjugate by bq, we check at once that it is enough to show

$$\mathfrak{b}_{w_0} = \sum_{w \in \mathfrak{C}_n} (\mathfrak{b}_{w_0} \cap \mathfrak{b}_{wb^{-1}})^{gr=w_0}.$$

Thus we reduce to show the following lemma :

Lemma 6.2. — *For all $i \geqslant j$, there exists $x_\ell \in k$ such that*

$$a^{i,j} := \delta_{i,j} + \sum_{\ell=j+1}^{i} x_\ell \delta_{\ell,j} \in (\mathfrak{b}_{w_0} \cap \mathfrak{b}_{c_{i,j}b^{-1}})^{gr=w_0}.$$

Proof. — We follow the proof of [**HMS22**]. For $i \geqslant j$, let $a^{i,j}$ be the element constructed at the beginning of the proof of Lemma 2.1 in *loc. cit.* For the convenience of the reader we recall its construction. Let $e_1, \dots, e_n$ be the standard basis of k^n and let $V_\bullet$ be the standard flag of k^n. Let $\mathcal{B}$ be the basis

$$b(e_1), b(e_2), \dots, b(e_{j-1}), e_j, b(e_{j+1}), \dots, b(e_i), e_{i+1}, \dots, e_n$$

of k^n. Then $a^{i,j}$ is the matrix, in the standard basis, of the endomorphism π of k^n defined by $\pi(x) = 0$ if $x \in \mathcal{B} \backslash \{e_j\}$ and $\pi(e_j) = e_i$. As in *loc. cit.* we check that

(i) $e_\ell \in \ker(\pi)$ if $\ell < j$ or $\ell > i$;
(ii) $\operatorname{Im}(\pi) \subset ke_i$ and $\pi(e_j) = e_i$;
(iii) the endomorphism $b^{-1}\pi b$ stabilizes the flag $c_{i,j}^{-1} V_\bullet$.

The first two points are checked in *loc. cit.* The third point follows from the fact that

$$b^{-1}\pi b(c_{i,j}^{-1} V_\ell) = \begin{cases} 0 & \text{if } \ell = 1, \dots, i-1 \\ kb^{-1}(e_i) & \text{if } \ell \geqslant i \end{cases} \tag{6}$$

and $kb^{-1}(e_i) \in V_i = c_{i,j}^{-1} V_i$. This shows that the matrix $a^{i,j}$ is an element of $\mathfrak{b}_{w_0} \cap \mathfrak{b}_{c_{i,j}b^{-1}}$ and has the form

$$\delta_{i,j} + \sum_{\ell=j+1}^{i} x_\ell \delta_{i,k} \tag{7}$$

for some $x_\ell \in k$.

It remains to check that $a^{i,j} \in (\mathfrak{b}_{w_0} \cap \mathfrak{b}_{c_{j,i}b^{-1}})^{gr=w_0}$ which is equivalent to check that the diagonal elements of $a^{i,j}$ and $\operatorname{Ad}(c_{i,j}b^{-1})a^{i,j}$ are the same. It follows from (6) that

$$(\operatorname{Ad}(c_{i,j}b^{-1})\pi)(e_\ell) = \begin{cases} 0 & \text{if } k < i \\ c_{i,j}b^{-1}(e_i) & \text{if } k = i \end{cases}$$

and $(\operatorname{Ad}(c_{i,j}b^{-1}\pi))(e_\ell) \in V_i$ if $\ell > i$. Therefore the diagonal of $\operatorname{Ad}(c_{i,j}b^{-1})a^{i,j}$ is zero except for the coefficient (i,i). On the other hand, we see by (7) that the diagonal entries of $a^{i,j}$ are all zero except (i,i). As the matrices $a^{i,j}$ and $\operatorname{Ad}(c_{i,j}b^{-1})a^{i,j}$ are conjugated they have the same trace and thus the same diagonal. □

□

7. Local deformation rings

The aim of this section is to prove Proposition 7.5. We are now in a purely local situation, and thus we freely use notations (in this section only) that were used before, hoping it will not lead to any confusion.

Let k be a field of characteristic 0. Let G be a split reductive group over k, $B \subset G$ a Borel subgroup of G, $T \subset B$ a maximal split torus of G, U the unipotent radical of

B and U^- the unipotent radical of the opposite Borel subgroup to B with respect to T (in particular $U^- \cap B = \{1\}$). Let $W := N_G(T)/T$ be the Weyl group of (G, T). Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{u}$, $\mathfrak{u}^-$ be the respective Lie algebras of G, B, T, U, U^-. Let $\tilde{\mathfrak{g}} \subset \mathfrak{g} \times G/B$ be the Grothendieck simultaneous resolution of $\mathfrak{g}$ and $X := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. We recall that X has irreducible components X_w which are indexed by the elements of the Weyl group W (see [**BHS19**, Def. 2.2.3]). The map $\tilde{\mathfrak{g}} \to \mathfrak{t}$ sending (ψ, gB) to the projection of $\mathrm{Ad}(g)^{-1}\psi$ on $\mathfrak{t}$ via $\mathfrak{b}/\mathfrak{u} \simeq \mathfrak{t}$ gives rise to two different maps $\kappa_1, \kappa_2 : X \to \mathfrak{t}$ corresponding to the two projections $X \to \tilde{\mathfrak{g}}$ and to a map $\kappa := (\kappa_1, \kappa_2) : X \to \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}$. If $w \in W$, we let $\mathfrak{t}^w \subset \mathfrak{t}$ be the subspace of elements fixed by w and $T_w \subset \mathfrak{t} \times_{\mathfrak{t}/W} \mathfrak{t}$ be the irreducible component

$$T_w := \{(x_1, x_2) \in \mathfrak{t} \times \mathfrak{t} \mid x_1 = \mathrm{Ad}(w)x_2\}.$$

The space X has a partition by locally closed subschemes V_w defined as inverse images of the Bruhat strata $U_w \subset G/B \times G/B$ by the map $\pi : X \to G/B \times G/B$ and $X_w = \overline{V_w}$. We have an inclusion $\kappa(X_w) \subset T_w$ ([**BHS19**, Lem. 2.5.1]).

Proposition 7.1. — *Let $x = (g_1B, 0, g_2B) \in X_{w_0}(k) \subset G(k)/B(k) \times \mathfrak{g} \times G(k)/B(k)$ be a k-point. Let $w \in W$ be such that $x \in V_w$ and assume that w_0w^{-1} is a product of distinct simple reflexions. Then we have an equality of k-vector spaces*

$$T_xX_{w_0} = T_x\kappa^{-1}(T_{w_0}).$$

Proof. — The inclusion $X_{w_0} \subset \kappa^{-1}(T_{w_0})$ induces an inclusion $T_xX_{w_0} \subset T_x\kappa^{-1}(T_{w_0})$. We will prove that these two k-vector spaces have the same dimension.

Let $k[\varepsilon] := k[X]/(X^2)$. The tangent space $T_x(\kappa^{-1}(t_{w_0}))$ is the set of $k[\varepsilon]$-points $(\tilde{g}_1B, \varepsilon A, \tilde{g}_2B)$ of X specialising to x such that moreover

$$\mathrm{Ad}(\tilde{g}_1)^{-1}(\varepsilon A) = \mathrm{Ad}(w_0)\,\mathrm{Ad}(\tilde{g}_2)^{-1}(\varepsilon A) \tag{8}$$

in $\mathfrak{t} \otimes_k k[\varepsilon]$. Let $\tilde{x} = (\tilde{g}_1B, \varepsilon A, \tilde{g}_2B)$ be such a point. We can write $\tilde{g}_i = g_i(1 + \varepsilon h_i)$, where $h_i \in \mathfrak{u}^-$. Using $\varepsilon^2 = 0$, the condition $\tilde{x} \in X(k[\varepsilon])$ is equivalent to $\mathrm{Ad}(g_i^{-1})A \in \mathfrak{b}$ for $i \in \{1, 2\}$. The condition (8) is then equivalent to $\mathrm{Ad}(g_1^{-1})A = \mathrm{Ad}(w_0)\,\mathrm{Ad}(g_2^{-1})A$ in $\mathfrak{t}$. Note that, up to changing x by a point of its $G(k)$-orbit, we can assume, without changing the dimensions of the tangent spaces, that $g_1 = 1$ and $g_2 = w$. The conditions above are then equivalent to

$$A \in \mathfrak{t}^{w_0w^{-1}} + (\mathfrak{u} \cap \mathrm{Ad}(w)\mathfrak{u})$$

which is a k-vector space of dimension $\dim_k \mathfrak{t}^{w_0w^{-1}} + \lg(w_0w^{-1})$. As w_0w^{-1} is a product of distinct simple reflexions, we have ([**Car72**, Lem. 2 & 3])

$$\dim_k \mathfrak{t}^{w_0w^{-1}} = \dim_k \mathfrak{t} - \lg(w_0w^{-1}).$$

Namely, we have a W-equivariant isomorphism $\mathfrak{t} \simeq \mathrm{Hom}(X^*(T), k)$, so that it is sufficient to prove that

$$\dim_{\mathbb{R}}(X^*(T) \otimes \mathbb{R})^{w'} = \dim_{\mathbb{R}}(X^*(T) \otimes \mathbb{R}) - \lg(w')$$

when w' is a product of simple reflexions. Let V be the subspace of $X^*(T) \otimes \mathbb{R}$ generated by the roots of (G, T). It is stable under W and has a direct summand on which W is acting trivially. It is therefore sufficient to prove that

$$\dim_{\mathbb{R}} V^{w'} = \dim_{\mathbb{R}} V - \lg(w').$$

As W is a finite group, $\dim_{\mathbb{R}} V - \dim_{\mathbb{R}} V^{w'}$ is equal to the number of eigenvalues of w' acting on V which are different from 1. By [**Car72**, Lem. 2], this number is equal to the number $l(w')$ of *loc. cit.* (which is a priori ***not*** $\lg(w')$). By [**Car72**, Lem. 3], we have $l(w') = \lg(w')$ since the set of simple roots is a set of linearly independent vectors of V.

Finally we deduce that

$$\dim_k T_x(\kappa^{-1}(T_{w_0})) \leqslant \dim_k(G/B \times G/B) + \dim_k \mathfrak{t} = \dim G.$$

On the other hand, we know that X_{w_0} is irreducible of dimension G. Consequently we have

$$\dim G \leqslant \dim_k T_x X_{w_0} \leqslant \dim_k T_x \kappa^{-1}(T_{w_0}) \leqslant \dim G$$

so that $T_x X_{w_0} = \dim_k T_x \kappa^{-1}(T_{w_0})$. □

Lemma 7.2. — ***Let $w \in W$ and $b \in B$. Then the point wB is in the closure of the T-orbit of bwB in G/B.***

Proof. — Let ν be a cocharacter of T such that $\langle \nu, \alpha \rangle > 0$ for all positive root α of (G, B, T). Then the map $\mathbb{G}_m \to G$ defined by $\nu u \nu^{-1}$ for u in the unipotent radical of B extends to a map $\mathbb{A}^1 \to G$ sending 0 to 1, thus as does the map $\nu b \nu^{-1}$. Consequently, as w normalises T, wB is in the closure of the image of the map

$$t \mapsto \nu(t) bwB = \nu(t) b \nu(t)^{-1} wB.$$

□

Lemma 7.3. — ***Let $(w_1, w_2) \in W^2$ and let $b \in B$. If $(w_1B, bw_2B) \in U_w \subset G/B \times G/B$, we have $w_1^{-1}w_2 \leqslant w$ in the Bruhat order.***

Proof. — If $t \in T$, we have $tw_1B = w_1B$ in G/B so that $(w_1B, tbw_2B) \in U_w$. It follows from Lemma 7.2 that w_2B is in the closure of the set $\{tbw_2B \mid t \in T\}$ so that (w_1B, w_2B) is in the closure if U_w. As the closure of U_w is the union of the U'_w with $w' \leqslant w$ and $(w_1B, w_2B) \in U_{w_1^{-1}w_2}$ we obtain the result. □

From now on we consider $K/\mathbb{Q}_p$ a finite extension, and denote $\Gamma = \mathrm{Gal}(K(\zeta_{p^\infty})/K)$. We fix L a finite extension of $\mathbb{Q}_p$ that splits K, i.e.

$$L \otimes_{\mathbb{Q}_p} K \simeq L^{[K:\mathbb{Q}_p]}.$$

We follow to the notation of [**KPX14**] concerning (φ, Γ)-modules over Robba rings. Let $\mathcal{R}(\pi_K)$ be the Robba ring for K (see [**KPX14**] definition 2.2.2). We define $t \in \mathcal{R}(\pi_K)$ by $t = \log(1 + \pi_K)$. Let $\mathcal{C}_L$ be the category local artinian $\mathcal{O}_L$-algebra A with maximal ideal $\mathfrak{m}_A$ such that the natural map $k_L \to A/\mathfrak{m}_A$ is an isomorphism. If A is an object of $\mathcal{C}_L$, we denote $\mathcal{R}_A(\pi_K) := A \otimes_{\mathbb{Q}_p} \mathcal{R}(\pi_K)$. We refer to [**KPX14**, Def. 2.2.12] for the notion of (φ, Γ)-module over $\mathcal{R}_A(\pi_K)$. Let D be a (φ, Γ)-module over $\mathcal{R}_L(\pi_K)$. We denote

$$\mathfrak{X}_D : \mathcal{C}_L \longrightarrow \mathrm{Sets}$$

the deformation functor of D, i.e. for an object A of $\mathcal{C}_L$, $\mathfrak{X}_D(A)$ is the set of isomorphism classes of pairs (D_A, i_A) where D_A is a (φ, Γ)-modules over $\mathcal{R}_A(\pi_K)$ and $i_A : L \otimes_A D_A \simeq D$ is an isomorphism of (φ, Γ)-modules. If (ρ, V) is a continuous representation of G_K on a finite dimensional L-vector space, the functor $\mathrm{D}_{\mathrm{rig}}$ of [**Ber02**] induces an isomorphism of deformation functors (see [**HMS22**, §3.6] for details)

$$\mathrm{D}_{\mathrm{rig}} : \mathfrak{X}_V \xrightarrow{\sim} \mathfrak{X}_{\mathrm{D}_{\mathrm{rig}}(V)}.$$

Let $F_\bullet = (\mathrm{Fil}_i\, D[t^{-1}])_{i\in\mathbb{Z}}$ be an increasing filtration of $D[t^{-1}]$ by sub-(φ,Γ)-modules over $\mathcal{R}_L(\pi_K)[t^{-1}]$ which are direct factors as $\mathcal{R}(\pi_K)[t^{-1}]$-modules, we define similarly

$$\mathfrak{X}_{D,F_\bullet} : \mathcal{C}_L \longrightarrow \mathrm{Sets}$$

the deformation functor of the pair $(D, F_\bullet)$, i.e. for A in $\mathcal{C}_L$, the set $\mathfrak{X}_{D,F_\bullet}(A)$ is the set of isomorphism classes of triples $(D_A, F_{\bullet,A}, i_A)$ where $(D_A, i_A) \in \mathfrak{X}_D(A)$ and $F_{\bullet,A}$ is a filtration of $D_A[t^{-1}]$ by (φ,Γ)-stable $\mathcal{R}_A(\pi_K)[t^{-1}]$-submodules which are direct factors of $D_A[t^{-1}]$ in the category of $\mathcal{R}_A(\pi_K)[t^{-1}]$-modules and such that $i_A(L \otimes_A F^i_{\bullet,A}) = F_i$ for all $i \in \mathbb{Z}$.

We recall some notations of [**BHS19**] section 3 and we refer the reader to *loc. cit.* for more precisions. Let W be an $L\otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$-representation of G_K which is almost de Rham. Let W^+ be a G_K-stable $L\otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}$-lattice of W. Let $\mathfrak{X}_W : \mathcal{C}_L \longrightarrow \mathrm{Sets}$ be the deformation functor of W, which means that $\mathfrak{X}_W(A)$ is the set of isomorphism classes of pairs (W^+_A, i_A) such that W^+_A is a finite free $A\otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}$-module endowed with a continuous semilinear action of G_K and i_A is a G_K-equivariant isomorphism $L\otimes_A W^+_A \simeq W^+$ of $L\otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}$-modules. If we fix an $L\otimes_{\mathbb{Q}_p} K$-linear isomorphism $\alpha : (L\otimes_{\mathbb{Q}_p} K)^n \xrightarrow{\sim} D_{\mathrm{pdR}}(W)$ we can define $\mathfrak{X}^\square_{W^+} : \mathcal{C}_L \longrightarrow \mathrm{Sets}$ the deformation functor of the pair (W^+, α). Let $\mathcal{F}_\bullet$ be a G_K-stable flag of $L\otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$-submodules of W, we define $\mathfrak{X}_{W^+,\mathcal{F}_\bullet}$ the deformation functor of the pair $(W^+, \mathcal{F}_\bullet)$ and $\mathfrak{X}^\square_{W^+,\mathcal{F}_\bullet}$ the deformation functor of the triple $(W^+, \mathcal{F}_\bullet, \alpha)$.

Now we fix $G = \mathrm{GL}_{n,K}$, $B \subset G$ the Borel subgroup of upper triangular matrices and $T \subset B$ the maximal torus of diagonal matrices. We recall that $\mathfrak{g}$ is the K-Lie algebra of G and $X = \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$. We also note $X_{K/\mathbb{Q}_p}$ and $\tilde{\mathfrak{g}}_{K/\mathbb{Q}_p}$ their Weil restrictions from K to $\mathbb{Q}_p$. If A is an object of $\mathcal{C}_L$ and $(W^+_A, \mathcal{F}_{\bullet,A}, \alpha_A)$ is an element of $\mathfrak{X}^\square_{W^+,\mathcal{F}_\bullet}(A)$, we can produce an element of $X_{K/\mathbb{Q}_p}(A)$ by sending $(W^+_A, \mathcal{F}_{\bullet,A}, \alpha_A)$ to $x_A := (\alpha^{-1}(\mathrm{D}_{\mathrm{pdR}}(\mathcal{F}_\bullet)), N_{W_A}, \alpha^{-1}(\mathrm{Fil}^\bullet_{W^+_A}))$. By [**BHS19**] Corollary 3.5.8, this map is a bijection. This implies that the functor $\mathfrak{X}^\square_{W^+,\mathcal{F}_\bullet}$ is pro-represented by the complete local ring of X at x_L.

Let $w \in W \simeq \mathfrak{S}_n^{[K:\mathbb{Q}_p]}$. Recall that $X_{K/\mathbb{Q}_p,w}$ is the irreducible component of $(X_{K/\mathbb{Q}_p})_L \simeq (X \times_K L)^{[L:K]}$ associated to w. Let D be a crystalline (φ,Γ)-module over $\mathcal{R}_L(\pi_K)$, together with a filtration $F_\bullet$ of $D[1/t]$. Let $W^+ = W^+_{\mathrm{dR}}(D)$, $W = W^+[t^{-1}]$ and $\mathcal{F}_\bullet = W_{\mathrm{dR}}(F_\bullet)$. As D is crystalline, the B_{dR}-representation W is de Rham and thus almost de Rham. The functors W_{dR} and W^+_{dR} induce a morphism of functors $\mathfrak{X}_{D,F_\bullet} \to \mathfrak{X}_{W^+,\mathcal{F}_\bullet}$. If D is moreover assumed to be φ-regular, this morphism if formally smooth by [**BHS19**, Cor. 3.5.4].

We define $\mathfrak{X}^{w,\square}_{W^+,\mathcal{F}_\bullet}$ as the subfunctor of $\mathfrak{X}^\square_{W^+,\mathcal{F}_\bullet}$ pro-represented by the quotient of $\widehat{\mathcal{O}}_{X_L,x_L}$ corresponding to the complete local ring of $X_{K/\mathbb{Q}_p,w}$ at x_L (with the convention that it is empty if $x_L \notin X_w$). We also define $\mathfrak{X}^w_{W^+,\mathcal{F}_\bullet} \subset \mathfrak{X}_{W^+,\mathcal{F}_\bullet}$ as the image of $\mathfrak{X}^{\square,w}_{W^+,\mathcal{F}_\bullet}$ via $\mathfrak{X}^\square_{W^+,\mathcal{F}_\bullet} \longrightarrow \mathfrak{X}_{W^+,\mathcal{F}_\bullet}$ and we define $\mathfrak{X}^w_{D,F_\bullet} \subset \mathfrak{X}_{D,F_\bullet}$ as the inverse image of $\mathfrak{X}^w_{W^+,\mathcal{F}_\bullet}$ by $\mathfrak{X}_{D,F_\bullet} \to \mathfrak{X}_{W^+,\mathcal{F}_\bullet}$.

We assume from now that D is crystalline and φ-generic (see [**HMS22**, §3.3]). Let $\mathfrak{X}^{\mathrm{cris}}_D \subset \mathfrak{X}_D$ be the subfunctor of crystalline deformations of D. Let $F_\bullet$ be a triangulation of D, we use the same symbol for the filtration induced on $D[1/t]$.

Lemma 7.4. — *We have a inclusion* $\mathfrak{X}_D^{\mathrm{cris}} \subset \mathfrak{X}_{D,F_\bullet}^{w_0}$.

Proof. — It follows from [**HMS22**, §3.3] that $\mathfrak{X}_D^{\mathrm{cris}} \subset \mathfrak{X}_{D,F_\bullet}$. We fix an isomorphism of $L \otimes_{\mathbb{Q}_p} K_0$-modules $\beta : (L \otimes_{\mathbb{Q}_p} K_0)^n \simeq D_{\mathrm{cris}}(D)$ such that $\beta \otimes \mathrm{Id}_K = \alpha$. Let $\mathfrak{X}_D^{\mathrm{cris},\square}$ be the functor of crystalline deformations of the pair (D, α). Let's consider the composite

$$\mathfrak{X}_D^{\mathrm{cris},\square} \longrightarrow \mathfrak{X}_{D,F_\bullet}^{\square} \longrightarrow \mathfrak{X}^{\square}_{W_{\mathrm{dR}}^+(D), W_{\mathrm{dR}}(F_\bullet)}.$$

Let $A \in \mathcal{C}_L$ and let $(D_A, \alpha_A) \in \mathfrak{X}_D^{\mathrm{cris},\square}(A)$ and let $(D_A, \alpha_A, F_{\bullet,A})$ be its image in $\mathfrak{X}_{D,F_\bullet}^{\square}(A)$. As D_A is crystalline, the operator ν_A on $W_{\mathrm{dR}}(D_A)$ is zero.

Now we remark that the schematic inverse image of $\{0\}$ by the natural map $(X_{K/\mathbb{Q}_p})_L \to (\mathfrak{g}_{K/\mathbb{Q}_p})_L$ of L-schemes is contained in the irreducible component $X_{K/\mathbb{Q}_p,w_0}$. Namely it is sufficient to prove the inverse image Z of $\{0\}$ by the natural map of K-schemes $X \to \mathfrak{g}$ is contained in X_{w_0}. But Z is $G/B \times \{0\} \times G/B$ which is the Zariski closure of $V_{w_0} \cap (G/B \times \{0\} \times G/B)$, so that $Z \subset X_{w_0}$.

This implies that the image of $(D_A, \alpha_A, F_{\bullet,A})$ in $\mathfrak{X}_{W_{\mathrm{dR}}^+(D), W_{\mathrm{dR}}(F_\bullet)}(A)$ is contained in $\mathfrak{X}^{w_0}_{W_{\mathrm{dR}}^+(D), W_{\mathrm{dR}}(F_\bullet)}(A)$ and finally that $\mathfrak{X}_D^{\mathrm{cris},\square} \subset \mathfrak{X}_{D,F_\bullet}^{w_0,\square}$ and $\mathfrak{X}_D^{\mathrm{cris}} \subset \mathfrak{X}_{D,F_\bullet}^{w_0}$. $\square$

We can now prove the main result of this section.

Proposition 7.5. — *Let D be a ϕ-generic, regular, crystalline, (φ, Γ)-module over $\mathcal{R}_L(\pi_K)$ and let $\mathrm{Tri}(D)$ be the set of triangulations of D. Then the following L-linear map is surjective:*

$$\bigoplus_{\mathcal{F} \in \mathrm{Tri}(D)} T\mathfrak{X}^{w_0}_{D,\mathcal{F}[1/t]} \longrightarrow T\mathfrak{X}_D.$$

Proof. — Let U be the kernel of the map $T\mathfrak{X}_D \to T\mathfrak{X}_{W_{\mathrm{dR}}^+(D)}$. It follows from Lemma 7.4 as in [**HMS22**, Cor. 3.13] that the following sequence is exact

$$0 \longrightarrow U \longrightarrow T\mathfrak{X}^{w_0}_{D,\mathcal{F}[\frac{1}{t}]} \longrightarrow T\mathfrak{X}^{w_0}_{W_{\mathrm{dR}}^+(D), W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \longrightarrow 0.$$

Therefore we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\bigoplus_{\mathcal{F}\in \mathrm{Tri}(D)} U \arrow[r, "\Sigma"] \arrow[d] & U \arrow[r] \arrow[d] & 0 \\
\bigoplus_{\mathcal{F}\in \mathrm{Tri}(D)} T\mathfrak{X}^{w_0}_{D,\mathcal{F}[\frac{1}{t}]} \arrow[r] \arrow[d] & T\mathfrak{X}_D \arrow[d, "W_{\mathrm{dR}}^+"] & \\
\bigoplus_{\mathcal{F}\in \mathrm{Tri}(\mathcal{D})} T\mathfrak{X}^{w_0}_{W_{\mathrm{dR}}^+(D), W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \arrow[r] \arrow[d] & T\mathfrak{X}_{W_{\mathrm{dR}}^+(D)} & \\
0 & &
\end{tikzcd}$$

Thus to prove that the middle horizontal arrow is surjective, it is sufficient to prove that the bottom horizontal arrow is surjective. As $\mathfrak{X}_D^\square \to \mathfrak{X}_D$ is formally smooth, it is sufficient to prove that the map

$$\bigoplus_{\mathcal{F}\in\mathrm{Tri}(\mathcal{D})} T\mathfrak{X}^{w_0,\square}_{W_{\mathrm{dR}}^+(D),W_{\mathrm{dR}}(\mathcal{F}[\frac{1}{t}])} \longrightarrow T\mathfrak{X}^\square_{W_{\mathrm{dR}}^+(D)}$$

is surjective. Let $F = W_{\mathrm{dR}}(\mathcal{F}[1/t])$ and $x_L \in X_{K/\mathbb{Q}_p}(L)$ be the corresponding point. It follows from [**BHS19**, Thm. 3.2.5 & Cor. 3.5.9] that the vertical arrows in the following commutative diagram are isomorphisms

$$\begin{tikzcd}
\mathfrak{X}^\square_{W^+,F} \arrow[r] \arrow[d, "\simeq"] & \mathfrak{X}^\square_{W^+} \arrow[d, "\simeq"] \\
(\widehat{X_{K/\mathbb{Q}_p,L}})_{x_L} \arrow[r, "\pi_2"] & (\widehat{\widetilde{\mathfrak{g}}}_{K/\mathbb{Q}_p,L})_{\pi_2(x_L)}
\end{tikzcd}$$

(π_2 is induced by the second projection $\widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$).

Recall that we have a decomposition $X_{K/\mathbb{Q}_p,L} \simeq \prod_{\tau\in\Sigma} X_\tau$ where $X_\tau \simeq L \times_{K,\tau} X$ and $\widetilde{\mathfrak{g}}_{K/\mathbb{Q}_p,L} \simeq \prod_{\tau\in\Sigma} \widetilde{\mathfrak{g}}_\tau$ and the map π_2 is of the form $(\pi_{2,\tau})_{\tau\in\Sigma}$ with $\pi_{2,\tau}$ the base change of the second projection $X \to \widetilde{\mathfrak{g}}$. Moreover the irreducible component of $X_{K/\mathbb{Q}_p,L}$ corresponding to the longest element is isomorphic to $\prod_{\tau\in\Sigma} X_{w_0,\tau}$ with w_0 the longest element of $\mathfrak{S}_n$.

Therefore we have to prove that the map

$$\bigoplus_{\tau\in\Sigma}\bigoplus_{\mathcal{F}\in\mathrm{Tri}(D)} \widehat{X}_{w_0,\tau,x_{L,\tau}} \longrightarrow \bigoplus_{\tau\in\Sigma} \widehat{\widetilde{\mathfrak{g}}}_{\tau,\pi_{2,L}(x_{L,\tau})}$$

is surjective at the level of tangent spaces. As the formation of tangent spaces commutes with finite products, it is sufficient to prove that for a fixed $\tau \in \mathrm{Tri}(D)$, the following map is surjective

$$\bigoplus_{\mathcal{F}\in\mathrm{Tri}(D)} \widehat{X}_{w_0,\tau,x_{L,\tau}} \longrightarrow \widehat{\widetilde{\mathfrak{g}}}_{\tau,x_{L,\tau}}.$$

Now up to change the basis α, we can assume that there is a non-critical triangulation $\mathcal{F}$ such that, for all $1 \leqslant i \leqslant n$, $L \otimes_{K,\tau} D_{\mathrm{pdR}}(\mathcal{F}_i)$ is generated by the first i vectors of the canonical basis. Thus its stabilizer is the standard Borel. Now by non criticality we can assume that the τ-part of the Hodge filtration is given by an element $h = bw_0 \in B(L)w_0$. Thus the previous surjectivity is equivalent to the following equality

$$\mathrm{Im}(\sum_{w\in\mathfrak{S}_n} T_{(wB(L),0,hB(L))}X_{w_0,\tau} \longrightarrow T_{(0,hB(L))}\widetilde{\mathfrak{g}}_\tau) = T_{(0,hB(L))}\widetilde{\mathfrak{g}}_\tau.$$

Let $\sigma \in W \simeq \mathfrak{S}_n$ be such that $(wB(L), 0, hB(L)) \in U_\sigma$. We claim that if $w = c_{i,j} := (i, i-1, \ldots, j)$, with $i \geqslant j$, then $w_0\sigma^{-1}$ is a product of distinct simple reflections. Namely it follows from Lemma 7.3 that $w^{-1}w_0 \leqslant \sigma$ so that $w_0\sigma^{-1} \leqslant w$ and, as w is a product of simple reflections, so is $w_0\sigma^{-1}$.

By Proposition 7.1, we deduce that

$$\begin{aligned} T_{(c_{i,j}B(L),0,hB(L))}X_{w_0,\tau} &= T_{(c_{i,j}B,0,hB)}\kappa^{-1}(T_{w_0}) \\ &= T_{c_{i,j}B(L)}G/B \oplus (c_{i,j}\mathfrak{b}c_{i,j}^{-1} \cap h\mathfrak{b}h^{-1})^{gr=w_0} \oplus T_{hB}G/B. \end{aligned}$$

Now we can use Lemma 6.1, and conclude that

$$\begin{aligned} &\mathrm{Im}\Big(\sum_{w\in\mathfrak{S}_n} T_{(wB,0,hB)}X_{w_0,\tau} \longrightarrow T_{(0,hB)}\widetilde{\mathfrak{g}}_\tau\Big) \\ &\qquad \supset \mathrm{Im}\Big(\sum_{i\geqslant j} T_{(c_{i,j}B,0,hB)}X_{\tau,w_0} \longrightarrow T_{(0,hB)}\widetilde{\mathfrak{g}}_\tau\Big) \\ &= \left(\sum_{i\geqslant j}(c_{i,j}\mathfrak{b}c_{i,j}^{-1} \cap h\mathfrak{b}h^{-1})^{gr=w_0}\right) \oplus T_{hB}G/B = h\mathfrak{b}h^{-1} \oplus T_{hB}G/B = T_{(0,hB)}\widetilde{\mathfrak{g}}_\tau. \qquad \square \end{aligned}$$

8. Global and local settings

Let $x \in \mathcal{X}^{mod,ag}$. Then x correspond to a cuspidal automorphic representation π of G. Let Π be the isobaric automorphic representation of $\mathrm{GL}_{n,E}$ associated to π by Theorem A.6.

Proposition 8.1. — *The representation Π is actually cuspidal and thus generic.*

Proof. — We have

$$\Pi = \Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_k$$

and a character χ_π where Π_i is a regular algebraic cuspidal automorphic representation of GL_{n_i} such that $\Pi_i \otimes \chi_\pi^{-1}$ is self-dual. Thus as $\rho_x = \rho_\pi$ is up-to-twist by a character given by $\rho_\Pi = \rho_{\Pi_1} \oplus \cdots \oplus \rho_{\Pi_k}$ but as x is in $X^{mod,ag}$, ρ_π is irreducible, thus $k = 1$ and Π is cuspidal. In particular it is generic by Piatieski-Shapiro, Shalika ([**CKM04**] Theorem 8.5). $\square$

Corollary 8.2. — *Let (ρ_x, V_x) be the representation corresponding to a point $x \in \mathcal{X}^{mod,ag}$. For all $v \in S$, $v \nmid p$, we have $H^0(G_v, \mathrm{ad}(V_x)^*(1)) = \{0\}$, in particular $H^1(G_v, \mathrm{ad}(V_x)) = H^1_f(G_v, \mathrm{ad}(V_x))$.*

Proof. — Let π be the automorphic representation associated to x. Since v is split in E, the representation $(\rho_x|_{G_v}, V_x)$ is a twist of the image of π_v by the local Langlands correspondence. By Proposition 8.1, the representation π_v is generic thus $H^0(G_v, \mathrm{ad}(V_x)^*(1)) = Hom_{G_v}(V_x, V_x(1)) = \{0\}$, thus $H^1(G_v, \mathrm{ad}(V_x))/H^1_f(G_v, \mathrm{ad}(V_x))$ vanishes too (for example [**Bel09**] Proposition 2.3 (i)). $\square$

Theorem 8.3 **(Newton-Thorne).** — *Let $x \in \mathcal{X}^{mod,ag}$ and let $r_x : G_{F,S} \to \mathcal{G}_n(\overline{\mathbb{Q}_p})$ be the associated representation. Then $H^1_f(G_{F,S}, \mathrm{ad}(r_x)) = \{0\}$.*

Proof. — This is consequence of [**NT19**, Thm. A]. Namely as $x \in \mathcal{X}^{mod,ag}$, the representation r_x is associated to an automorphic representation π whose base change to $\mathrm{GL}_n(\mathbb{A}_E)$ is cuspidal algebraic and regular by Proposition 8.1. $\square$

Let $x \in \mathcal{X}^{mod,ag}$ and let (ρ_x, V_x) be the associated representation of $G_{E,S}$ over a finite extension of $k(x)$. Our goal is to prove Theorem 8.7, which is invariant by scalar extension, thus we freely extend scalars of $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ to assume ρ_x is defined over $k(x)$. Let $\mathcal{F}$ be a refinement of (ρ_x, V_x), that is, a family $(\mathcal{F}_v)_{v\in S_p}$ where $\mathcal{F}_v$ is a refinement of the crystalline representation $(\rho_x|_{G_v}, V_x)$. Let $x_{\mathcal{F}} \in \mathcal{E}$ be the classical dominant point corresponding to ρ_x and the refinement $\mathcal{F}$. If what follows, if X is a rigid space and $x \in X$, we set $\widehat{X}_x := \operatorname{Spec}(\widehat{\mathcal{O}_{X,x}})$. The projection map $\mathcal{E} \to \mathcal{X}_{\overline{\rho}}$ induces a morphism $\widehat{\mathcal{E}}_{x_{\mathcal{F}}} \to \left(\widehat{\mathcal{X}_{\overline{\rho}}^{\chi-pol}}\right)_x \simeq \mathfrak{X}_{\rho_x}^{\chi-pol}$. For each $v \in S_p$, let $\rho_{x,v} = \rho_x|_{G_v}$, which is irreducible, and consider the composite map

$$\widehat{\mathcal{E}_{x_{\mathcal{F}}}} \longrightarrow \left(\widehat{\mathcal{X}_{\overline{\rho}}^{\chi-pol}}\right)_x \simeq \mathfrak{X}_{\rho_x}^{\chi-pol} \longrightarrow \mathfrak{X}_{\rho_{x,v}}.$$

In the previous section, and in terms of φ, Γ-modules, we have defined deformation spaces $\mathfrak{X}_{\rho_v,\mathcal{F}}^{qtri,w} := \mathfrak{X}_{D_{rig}(\rho_v),\mathcal{F}[1/t]}^{w}$ which we call *quasi-trianguline*. Denote also $\mathfrak{X}_{\rho_v}^{cris} := \mathfrak{X}_{D_{rig}(\rho_v)}^{cris}$ the crystalline deformation ring (see [**Che11**, page 24] or [**HMS22**, page 18]).

Lemma 8.4. — *The map $\widehat{\mathcal{E}_{x_{\mathcal{F}}}} \to \mathfrak{X}_{\rho_{x,v}}$ factors through $\mathfrak{X}_{\rho_{x,v},\mathcal{F}_v}^{qtri,w_0}$.*

Proof. — Let $\overline{\rho}_v : G_{F_v} \to \mathrm{GL}_n(k)$ be the composite of $\overline{\rho}$ with $G_{F_v} \hookrightarrow G_{E,S}$ and let $\mathfrak{X}_{\overline{\rho}_v}^{\square}$ be the framed deformation space of $\overline{\rho}_v$. Let $X_{\mathrm{tri}}(\overline{\rho}_v) \subset \mathfrak{X}_{\overline{\rho}_v}^{\square,\mathrm{rig}} \times \widehat{F_v^{\times}}^n$ be the trianguline variety. We choose $y \in \mathcal{X}_{\overline{\rho}_v}^{\square} := \mathfrak{X}_{\overline{\rho}_v}^{\square,\mathrm{rig}}$ be a point such that ρ_y is conjugated to $\rho_{x,v}$ and let $y_{\mathcal{F}_v}$ be the dominant point of $X_{\mathrm{tri}}(\overline{\rho}_v)$ corresponding to y and to the refinement $\mathcal{F}_v$. The projection map $X_{\mathrm{tri}}(\overline{\rho}_v) \to \mathfrak{X}_{\overline{\rho}_v}^{\square,\mathrm{rig}}$ induces a map $\widehat{X_{\mathrm{tri}}(\overline{\rho}_v)}_{y_{\mathcal{F}_v}} \to \mathfrak{X}_{\rho_y}^{\square}$ and, by [**BHS19**, Cor. 3.7.8], this morphism factors through $\mathfrak{X}_{\rho_y,\mathcal{F}_v}^{\square,w_0}$. As $\mathfrak{X}_{\rho_y,\mathcal{F}_v}^{\square,w_0}$ is the pullback of $\mathfrak{X}_{\rho_y,\mathcal{F}_v}^{w_0}$ by the formally smooth map $\mathfrak{X}_{\rho_y}^{\square} \to \mathfrak{X}_{\rho_y}$, it is sufficient to prove that there exists, locally at x, a factorization

$$\begin{tikzcd} \mathcal{E} \arrow[r, dotted] \arrow[dr] & X_{\mathrm{tri}}(\overline{\rho}_v) \arrow[r] \arrow[d] & \mathcal{X}_{\overline{\rho}_v}^{\square} \arrow[dl] \\ & \mathcal{X}_{\overline{\rho}_v} & \end{tikzcd}$$

sending $x_{\mathcal{F}}$ on $y_{\mathcal{F}_v}$, where $\mathcal{X}_{\overline{\rho}_v}$ is the rigid fiber of the pseudo-deformation space, as in Definition 3.2. As $\rho_{x_{\mathcal{F}}}$ is irreducible, it follows that there exists some affinoïd neighborhood U of $x_{\mathcal{F}}$ in $\mathcal{E}$ and a continuous morphism $\rho_V : G_{E,S} \to \mathrm{GL}_n(\mathcal{O}(U))$ such that $\mathrm{Tr}(\rho_U)(z) = \mathrm{Tr}(\rho_z)$ for all $z \in U$. Indeed, by [**Che14**] Theorem 2.22 there is a representation $\rho_A : G_{E,S} \to \mathrm{GL}_n(\mathcal{O}_{\mathcal{E},x_{\mathcal{F}}})$ whose trace is $D_{|\mathcal{O}_{\mathcal{E},x_{\mathcal{F}}}}$. As $\mathcal{O}_{\mathcal{E},x_{\mathcal{F}}}$ is a direct limit over U, there exists such a U (see [**BC09**] Lemma 4.3.7 for a precise argument). This gives us a map $U \to \mathcal{X}_{\overline{\rho}_v}^{\square}$ and even $U \to \mathcal{X}_{\overline{\rho}_v}^{\square} \times \widehat{F_v^{\times}}^n$. As the set $\mathcal{Z}'$ is Zariski-dense and accumulation in $\mathcal{E}$, we can choose U so that $U \cap \mathcal{Z}'$ is Zariski-dense in U. A point of $U \cap \mathcal{Z}'$ is sent to a point of $X_{\mathrm{tri}}(\overline{\rho})$ by $U \to \mathcal{X}_{\overline{\rho}_v}^{\square} \times \widehat{F_v^{\times}}^n$ (by definition of $X_{\mathrm{tri}}(\overline{\rho})$, [**BHS19**] section 3.7) so that we obtain the desired section. $\square$

For a global representation ρ of G_F, or a polarised representation of G_E, $\rho_p = (\rho_v)_{v|\,p}$ where $v|\,p$ in F. Then we write $\mathfrak{X}^{qtri,w}_{\rho_p,\mathcal{F}} := \prod_{v|p} \mathfrak{X}^{qtri,w}_{\rho_v,\mathcal{F}_v}$ and $\mathfrak{X}^{cris}_{\rho_p} := \prod_{v|p} \mathfrak{X}^{cris}_{\rho_v}$.

Definition 8.5. — Let $x = (\rho_p, \mathcal{F})$ be a local representation of G_K, with $\mathcal{F}$ a (quasi-) triangulation of ρ_p i.e. of $D_{rig}(\rho_p)[t^{-1}]$. We associate to $x = (\rho_p, \mathcal{F})$ a permutation $w_x \in \mathfrak{S}_n^{[K:\mathbb{Q}_p]}$ measuring the relative position of $\mathcal{F}$ and the Hodge filtration of ρ_p (see [**BHS19**] before Proposition 3.6.4). We say that x, or $\mathcal{F}$, is associated to a product of distinct transpositions if w_x is a product of distinct *simple* transpositions.

The following corollary is very similar to [**Ber20**] and [**BHS19**].

Corollary 8.6. — *For $x \in \mathcal{X}^{mod,ag}$ as before, and a refinement $\mathcal{F}$, which is associated to a product of distinct transpositions. Then x_F is a smooth point of $\mathcal{E}$ and we have an isomorphism*

$$T_{x_\mathcal{F}}\mathcal{E} \xrightarrow{\sim} T\mathfrak{X}^{qtri,w_0}_{\rho_{x,p},\mathcal{F}}/T\mathfrak{X}^{cris}_{\rho_{x,p}}.$$

Proof. — Denote by $\mathfrak{X}^{\chi-pol}_{\rho_x}$ the (equicharacteristic) χ-polarised global deformation space of $\mathrm{tr}\,\rho_x$. It is the completion of $\mathcal{X}^{\chi-pol}_{\overline{\rho}}$ at ρ_x by [**Che14**, section 4.1]. Denote by $\mathfrak{X}^{\mathrm{tri}}_{\rho_x,\mathcal{F}}$ the fiber product $\mathfrak{X}^{\chi-pol}_{\rho_x} \times_{\mathfrak{X}_{\rho_{x,p}}} \mathfrak{X}^{qtri,w_0}_{\rho_{x,p},\mathcal{F}}$. We have a map

$$\widehat{\mathcal{E}_{x_\mathcal{F}}} \longrightarrow \mathfrak{X}^{tri}_{\rho_x,\mathcal{F}},$$

induced from the map $\widehat{\mathcal{E}_{x_\mathcal{F}}} \longrightarrow \mathfrak{X}^{qtri,w_0}_{\rho_{x,v},\mathcal{F}_v}$ and $\widehat{\mathcal{E}_{x,\mathcal{F}}} \longrightarrow \mathfrak{X}^{\chi-pol}_{\rho_x}$. But then the standard argument that

$$f : \mathcal{O}_{\mathfrak{X}^{\mathrm{tri}}_{\rho_x,\mathcal{F}}} \longrightarrow \widehat{\mathcal{O}_{\mathcal{E}_{x,\mathcal{F}}}},$$

is surjective comes from the fact that $\mathcal{E}_{x_\mathcal{F}}$ is topologically generated by $\mathcal{O}_{\mathfrak{X}^{\chi-pol}_{\rho_x}}$ and $\mathcal{O}_\mathcal{T}$ by construction, but $\mathfrak{X}^{qtri,w_0}_{\rho_{x,p},\mathcal{F}}$ lies over $\mathcal{T}$. Thus we have a closed immersion

$$\widehat{\mathcal{E}_{x,\mathcal{F}}} \hookrightarrow \mathfrak{X}^{tri}_{\rho_x,\mathcal{F}}.$$

But the genericity assumption (Corollary 8.2) implies that the tangent space of $\mathfrak{X}^{tri}_{\rho_x,\mathcal{F}}$ sits inside

$$H^1_{f'}(G_F, ad(\rho_x)) := \ker\left(H^1(G_F, ad(\rho_x)) \longrightarrow \prod_{v\nmid p} H^1(G_{F_v}, ad(\rho_x))/H^1_f(G_{F_v}, ad(\rho_x))\right),$$

thus we have an the exact sequence

$$0 \longrightarrow H^1_f(G_F, ad(\rho_x)) \longrightarrow T\mathfrak{X}^{tri}_{\rho_x,\mathcal{F}} \longrightarrow \bigoplus_{v|p} T\mathfrak{X}^{qtri,w_0}_{\rho_{x,v},\mathcal{F}_v}/T\mathfrak{X}^{cris}_{\rho_{x,v}}.$$

Moreover, because of the previous surjection, we have inequalities

$$\dim T_{(x,\mathcal{F})}\mathcal{E} \leqslant \dim T\mathfrak{X}^{tri}_{\rho_x,\mathcal{F}} \leqslant n[F:\mathbb{Q}],$$

where the last inequality is [**BHS19**] Proposition 4.1.5 (together with Remark 4.1.6 (ii) and Corollary 3.7.8) as $\mathcal{F}$ is a product of distinct transpositions, and the Theorem 8.3 of Newton-Thorne which assures that

$$H^1_f(G_F, ad(\rho_x)) = \{0\}.$$

But as $\mathcal{E}$ is equidimensional of dimension $n[F:\mathbb{Q}]$, we have $\dim T_{x_{\mathcal{F}}}\mathcal{E} = n[F:\mathbb{Q}]$ and thus $x_{\mathcal{F}}$ is a smooth point of $\mathcal{E}$ and we have

$$\widehat{\mathcal{E}_{x_{\mathcal{F}}}} \xrightarrow{\sim} \mathfrak{X}^{tri}_{\rho_x,\mathcal{F}},$$

and thus

$$T_{x_{\mathcal{F}}}\mathcal{E} \simeq T\mathfrak{X}^{tri}_{\rho_x,\mathcal{F}} \simeq \bigoplus_{v|p} T\mathfrak{X}^{qtri,w_0}_{\rho_{x,v},\mathcal{F}_v}/T\mathfrak{X}^{cris}_{\rho_{x,v}}. \qquad \square$$

Theorem 8.7. — *For $x \in \mathcal{X}^{mod,ag}$, the image of the natural map*

$$\bigoplus_{\mathcal{F}} T_{x_{\mathcal{F}}}\mathcal{E} \longrightarrow T_x\mathcal{X}^{\chi-pol}_{\overline{\rho}},$$

has dimension at least $\frac{n(n+1)}{2}[F:\mathbb{Q}]$, where $\mathcal{F}$ runs over the $n![F:\mathbb{Q}]$ (classical) refinements of x.

Proof. — Indeed, for each $\mathcal{F}$ of the form $(c^v_{i,j}\cdot\mathcal{F}_{v,0})_v$ as in the proof of Proposition 7.5, we have,

$$T_{x_{\mathcal{F}}}\mathcal{E} \simeq T\mathfrak{X}^{qtri,w_0}_{\rho_{x,p},\mathcal{F}}/T\mathfrak{X}^{crys}_{\rho_{x,p}},$$

and moreover, for all $v|p$, the map

$$\bigoplus_{i,j} T\mathfrak{X}^{qtri,w_0}_{\rho_{x,v},c^v_{i,j}\mathcal{F}_{v,0}} \longrightarrow T\mathfrak{X}_{\rho_v},$$

is surjective by Proposition 7.5. In particular, the map

$$\bigoplus_{v,c^v_{i,j}} T_{(x,c^v_{i,j}\cdot\mathcal{F}_{v,0})}\mathcal{E} \longrightarrow T_x\mathcal{X}^{\chi-pol}_{\overline{\rho}} = T\mathfrak{X}^{\chi-pol}_{\rho_x} \longrightarrow T\mathfrak{X}_{\rho_p}/T\mathfrak{X}^{crys}_{\rho_p},$$

which can also be factored,

$$\bigoplus_{v,c^v_{i,j}} T_{(x,c^v_{i,j}\cdot\mathcal{F}_{v,0})}\mathcal{E} \longrightarrow \bigoplus_{v|p}\bigoplus_{c^v_{i,j}} T\mathfrak{X}^{qtri,w_0}_{\rho_{x,v},c^v_{i,j}\mathcal{F}_{v,0}}/T\mathfrak{X}^{crys}_{\rho_{x,v}} \xrightarrow{\Sigma_{\mathcal{F}}} T\mathfrak{X}_{\rho_p}/T\mathfrak{X}^{crys}_{\rho_p},$$

is surjective by Corollary 8.6 and Proposition 7.5, and thus has rank at least $\frac{n(n+1)}{2}[F:\mathbb{Q}]$, thus the same is true for the map,

$$\bigoplus_{\mathcal{F}} T_{(x,\mathcal{F})}\mathcal{E} \longrightarrow T_x\mathcal{X}^{\chi-pol}_{\overline{\rho}}. \qquad \square$$

Remark 8.8. — Note that we don't actually need all the refinements (for a fixed v), only the $1+\frac{n(n-1)}{2}$ refinements given by $c_{i,j} = (i, i-1, \dots, j) \in \mathfrak{S}_n$ with $i \geqslant j$ (starting from a non-critical one). But this is still more than just the n well-positioned refinements for weakly generic points of Chenevier **[Che11]**, even for $n = 3$.

Theorem 8.9. — *Let $\overline{\mathcal{F}(\overline{\rho})} \subset \mathcal{X}^{\chi-pol}_{\overline{\rho}}$ be the Zariski closure of the image of $\mathcal{E}(\overline{\rho})$. Then $\overline{\mathcal{F}(\overline{\rho})}$ is equidimensional of dimension $\frac{n(n+1)}{2}[F:\mathbb{Q}]$, and is a union of irreducible components of $\mathcal{X}^{\chi-pol}_{\overline{\rho}}$.*

Proof. — We have already proven that almost generic point are Zariski dense in $\overline{\mathcal{F}(\overline{\rho})}$ (see Theorem 5.8). It is enough to prove that these points are smooth points of $\mathfrak{X}_{\overline{\rho}}$ whose local ring are $\frac{n(n+1)}{2}[F:\mathbb{Q}]$-dimensional. This is essentially Allen's proof [**All16**]. Let x be such a almost generic point in $\mathcal{F}(\overline{\rho})$, thus ρ_x is irreducible and we can consider the polarised deformation space $\mathfrak{X}^{\chi-pol}_{\rho_x}$. Then by an argument of Kisin (see also [**All16**]),

$$\mathfrak{X}^{\chi-pol}_{\rho_x} \simeq \widehat{(\mathfrak{X}^{\chi-pol}_{\overline{\rho}})}_x.$$

Thus we need to show that $\mathfrak{X}^{\chi-pol}_{\rho_x}$ is (formally) smooth of dimension $\frac{n(n+1)}{2}[F:\mathbb{Q}]$. But as ρ_x is absolutely irreducible we can choose a lift r_x to $\mathcal{G}_n$ and by Proposition 3.6 reduce to $\mathfrak{X}^{\chi}_{r_x}$. Remark here that because of Proposition 8.1 and Theorem 2.3, we can apply Proposition 3.4. Calculations on the dimension of deformation ring made in Proposition 3.4 show that we are thus reduced to show that $h^2(G_{F,S}, ad(r_x)) = 0$, or what is equivalent $h^1(G_{F,S}, ad(r_x)) = \frac{n(n+1)}{2}[F:\mathbb{Q}]$. But as ρ_x is generic at p by Proposition 8.1, by remark 1.2.9 of [**All16**], we get $H^1_g(G_{F,S}, \mathrm{ad}'(\rho_x)) = H^1_f(G_{F,S}, \mathrm{ad}'(\rho_x))$ which vanishes by Newton-Thorne's Theorem 8.3. Thus the following map is injective,

$$H^1(G_{F,S}, \mathrm{ad}'(\rho_x)) \longrightarrow \prod_{v|p} H^1(F_v, \mathrm{ad}'(\rho_x))/H^1_g(\mathrm{ad}'(\rho_x)).$$

But then we prove exactly as in [**All16**], Lemma 1.3.5, as our x is HT-regular, that the space $\mathfrak{X}^{\chi}_{r_x}$ is formally smooth of dimension $\frac{n(n+1)}{2}[F:\mathbb{Q}]$, thus by Proposition 3.4 $\mathfrak{X}^{\chi-pol}_{\rho_x}$ is formally smooth of dimension $\frac{n(n+1)}{2}[F:\mathbb{Q}]$, but as it contains the local ring of the closure of $\mathcal{F}(\overline{\rho})$ at ρ_x, which is of dimension $\geqslant \frac{n(n+1)}{2}[F:\mathbb{Q}]$, both local rings are equal (and $\mathcal{F}(\overline{\rho})$ is smooth at these points). □

Remark 8.10. — Recall that in the previous theorem $\mathcal{E}(\overline{\rho})$ and thus $\overline{\mathcal{F}(\overline{\rho})}$ depend on the choice of an auxiliary level K^p outside p. We can ask how the closure of the infinite fern depends on K^p. If we let K^p appear in the notations, we can at least have an optimal K^p.

Corollary 8.11. — *There exists a level K^p outside p, such that for all level $K^{',p}$ outside p, the Zariski closure of the infinite fern of tame level K^p, $\overline{\mathcal{F}_{K^p}(\overline{\rho})}$, contains the infinite fern of level $K^{',p}$, $\overline{\mathcal{F}_{K^{',p}}(\overline{\rho})}$.*

Proof. — As $\mathcal{X}^{\chi-pol}_{\overline{\rho}}$ is the generic fiber of a noetherian excellent formal scheme, it has a finite number of connected component (See [**Con99**, Theorem 2.3.1]) . Thus as the number of components in the closure of the infinite fern (by Theorem 8.9) grows with K^p, it eventually stabilizes. □

We can now deduce the following corollary, which is due to Allen, [**All19**], for which we need to take care that automorphic points given by [**All19**] main's theorem are indeed inside our infinite fern. So we assume the following,

Hypothesis 8.12. —
1. $p > 2$, is unramified in E and every prime $v|p$ in F splits in E. Moreover, $\zeta_p \notin E$.
2. $\overline{\rho}(G_{E(\zeta_p)})$ is adequate, $\overline{\rho}$ is polarized by χ i.e. $\overline{\rho}^\vee \simeq \overline{\rho}^c \otimes \chi\varepsilon^{n-1}$.

3. There exists a GL_n-automorphic representation Π_0, which is regular algebraic χ-polarized cuspidal, such that ρ_π lifts $\overline{\rho}$ and such that $\rho_{\pi,v}$ is potentially diagonalisable for all $v|p$, and even ordinary for all $v|p$ if $p|n$.
4. χ is crystalline at p, satisfies $\chi = \chi^c$ and satisfies a sign condition (see Hypothesis 3.7 and section 2)
5. $H^0(G_v, \mathrm{ad}(\overline{\rho})(1)) = 0$ for all $v|p$.

We still hope that hypothesis 1. and 4. are technical and we hope to be able to remove then, as for Theorem 8.9. It is unkown at the moment if all potentially crystalline representations are potentially diagonalisable, i.e. if we could relax hypothesis 3. to a classical modularity (for GL_n, crystalline at p say). We hope that hypothesis 2. is unnecessary, but at the moment the main result of [**All19**] relies on it, and also on 5. but we imagine that it could be removed using new results on local deformations rings (e.g. [**BIP21**]).

We have the following

***Corollary 8.13* (Allen).** — *Assume the hypothesis 8.12. Then the generic fiber of the global deformation ring,* $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$*, is equidimensional of dimension* $[F:\mathbb{Q}]\frac{n(n+1)}{2}$ *and the infinite fern* $\mathcal{F}(\overline{\rho})$ *is Zariski dense in* $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ *thus in* $\mathrm{Spec}(R_{\overline{\rho}}^{\chi-pol})$*. In particular automorphic points are dense in* $\mathrm{Spec}(R_{\overline{\rho}}^{\chi-pol})$.

Proof. — As the set of Hypothesis 8.12 contains strictly the hypothesis of Theorem 8.9, we have that the Zariski closure of $\mathcal{F}(\overline{\rho})$ (if non-empty!) is a union of connected components of $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$. Thus, it is enough to prove that each component of $\mathcal{X}_{\overline{\rho}}^{\chi-pol}$ contains a points in the infinite fern, and by the reduction of Lemma 3.9 and considering $\Pi_0\psi_0^{-1}$ where ψ_0 given by [**CHT08**, Lem.4.1.5] , we can assume $\chi = 1$. By [**All19**], Corollary 5.3.3, we have that $R_{\overline{\rho}}^{pol}$ is $\mathcal{O}$-flat, reduced, and complete intersection of the expected dimension, but we still need to check that the automorphic point in all components can be chosen to be in the infinite fern (i.e. holomorphic at infinity automorphic representations for GU). Let $\mathcal{C}$ be an irreducible component of $\mathcal{X}_{\overline{\rho}}^{pol}$, which is of the form $\mathcal{C} = C^{rig}$ for an irreducible component C of $\mathrm{Spec}(R_{\overline{\rho}}^{pol})$ ([**All19**, Lemma 1.2.3]). By [**All19**, Theorem 5.3.1,Theorem 5.3.2], there is a GL_n-automorphic cuspidal point Π in $\mathcal{C}$, which is moreover unramified at places above p, very regular, self dual, and such that Π is a smooth point of $\mathfrak{X}_{\overline{\rho}}^{pol}$ (see [**All16**] Theorem C). In particular, by [**Mok15**], there exists π_0 a cuspidal, regular algebraic, unramified above p, representation of the quasi-split unitary group U whose base change is Π. By [**Mok15**] again, to Π which is conjugate self dual thus can be seen as an Arthur parameter, is associated a global A-packet Π_U for U which contains π_0. Let ψ_∞ be the associated archimedean Arthur parameter at any place $v|\infty$ of F. It is a tempered parameter as Π_U is, and $\psi_{\infty|\mathbb{C}^\times}$ can be assumed of the form

$$z \longmapsto (z^{a_1}\overline{z}^{b_1}, \dots, z^{a_n}\overline{z}^{b_n}),$$

and for $j \in W_{\mathbb{R}}\backslash\mathbb{C}^\times$ such that $jzj^{-1} = \overline{z}$, we can check that this implies that this is conjugate to $(z^{-b_1}\overline{z}^{-a_1}, \dots, z^{-b_n}\overline{z}^{-a_n})$. In particular, as the weight is regular this implies that there exists $\sigma \in \mathfrak{S}_n$ an involution such that $a_{\sigma(i)} = -b_i$. But as Π is regular, algebraic, cuspidal, by Clozel's purity Lemma, we have $a_i + b_i = 0$ for all i (which in particular implies that ψ_∞ is bounded on $W_{\mathbb{R}}$). Thus, actually $\sigma = 1$. We now check that ψ_∞ is indeed discrete. As in [**BC05**] Lemme 4.3.1 we have, writing $\psi_\infty(j) = h \rtimes c$, we have for $z \in \mathbb{C}^\times$,

$$\psi_\infty(\overline{z}) = \psi_\infty(jzj^{-1}) = hw_0{}^t\psi_\infty(z)^{-1}w_0^{-1}h^{-1} \rtimes 1.$$

Thus h normalizes the torus and by regularity and using $a_i = -b_i$ we must have that h is up to conjugacy w_0^{-1}, thus the parameter ψ_∞ is discrete. As ψ_∞ is tempered, discrete, regular algebraic, it is equal to the L-packet of discrete series representations constructed in [**Lan89**], and in particular contains the holomorphic discrete series. Thus we can change π_0 by π_1 still in Π_U whose component at infinity is holomorphic. As Π is cuspidal, the A-packet Π_U is stable and thus $\mathcal{S}_\psi = 1$, so that π_1 is also discrete and automorphic. In other words we can assume that $\pi_0 = \pi_1$ is holomorphic at infinity and unramified at p.

Choose an algebraic extension of its central character which is unramified at p, then by [**LS19**] there exists an extension of π_0 to a cuspidal, regular algebraic representation π of GU. Moreover π_∞ is also the holomorphic discrete series thus contribute to coherent cohomology in degree 0 and thus gives a point in the Eigenvariety $\mathcal{E}$ (for GU), whose Galois representation (given in Corollary A.8) is ρ. In particular, $\mathcal{F}(\overline{\rho})$ intersect $\mathcal{C}$, and the corollary is proved ! $\square$

Appendix A. Similitude Unitary groups, Tori, Base Change and Galois representations

Denote by $G = GU(V)$ a similitude unitary group over $\mathbb{Q}$ (with similitude factor in $\mathbb{Q}^\times$) associated to the CM extension E/F, and by $Z \simeq GU(E)$ its center.

Let ℓ be a rational prime, unramified in E, which is also unramified for $GU(V)$ (i.e. $GU(V)_{\mathbb{Q}_p}$ is quasi-split, and split over an unramified extension). Let π be a cuspidal automorphic representation of G, assume π is unramified at ℓ, and choose a maximal compact K at ℓ for which π is unramified. Then π_ℓ^K is a 1-dimensional representation of $H_{\mathbb{C}}(G(\mathbb{Q}_\ell), K)$, the Hecke algebra of bi-K-invariant $\mathbb{C}$-valued functions on $G(\mathbb{Q}_\ell)$ with compact support. The Satake isomorphism and the unramified local Langlands correspondance ([**Bor79**]) associate to it an unramified representation with values in the L-group of G (actually in the L-group of T). Denote T_U, T the maximal torus of $U = U(V)$ and G respectively. The natural inclusion $T_U \subset T$, which is compatible with the Galois action and central, gives a map

$$ {}^LT \longrightarrow {}^L T_U. \tag{9}$$

Proposition A.1. — *To π_ℓ we can associate an unramified Langlands parameter*

$$r : W_{\mathbb{Q}_\ell} \longrightarrow {}^LT_U.$$

For all $\lambda | \ \ell$ in F and $\lambda' | \ \lambda$ in E, we can restrict r to $W_{E'_\lambda}$ and then compose

$$r : W_{E_{\lambda'}} \longrightarrow {}^LT_{GL_n} =: \widehat{\mathbb{G}_m^n} \times W_{E_{\lambda'}}.$$

This induces a well-defined class (up to conjugacy), for all $\lambda' | \ell$,

$$r_{\lambda'} : W_{E'_\lambda} \longrightarrow \mathrm{GL}_n(\mathbb{C}).$$

Proof. — As ℓ is unramified for E and G (thus U), actually $U_{\mathbb{Q}_p}$ is isomorphic to $U(n)_{E/F,\mathbb{Q}_p}$ for any choice of (unramified) unitary group of rank n, so choose the one with anti-diagonal matrix form. With this form, we check that actually the upper triangular Borel is indeed a Borel over $\mathbb{Q}_p$, and its maximal torus is the diagonal one, given by

$$T_U = \{Diag(a_1, \dots, a_n) | a_i \in E^\times, c(a_i)a_{n+1-i} = 1\} \subset T_{GL_{n,E}},$$

with $c \in \mathrm{Gal}(E/F)$ the complex conjugacy. Denote by Σ_E the complex embeddings of E. We then have that its characters are given by $X^*(T)$ a quotient of $(\mathbb{Z}^n)^{\Sigma_E}(= X^*(T_{GL_{n,E}}))$ by the relation $(\lambda_{i,\sigma})_{i,\sigma} = (\lambda^{-1}_{n+1-i,\sigma c})_{i,\sigma}$. Its cocharacters are given by $X_*(T) \subset (\mathbb{Z}^n)^{\Sigma_E} = X_*(T_{GL_{n,E}})$ such that the collection $(\mu_{i,\sigma})_{i,\sigma}$ satisfies $\mu_{i,\sigma} = \mu^{-1}_{n+1-i,\sigma c}$. The Galois action $\sigma \in G_{\mathbb{Q}}$ sends the character $\lambda_{\tau,i}$ to $\sigma \cdot \lambda_{\tau,i} := \lambda_{\tau,i} \circ \sigma^{-1} = \lambda_{\sigma\tau,i}$. It sends the cocharacter $\mu_{\tau,i}$ to $\sigma \circ \mu_{\tau,i} = \mu_{\sigma\tau,i}$. Then the dual torus is given by the subtorus of $\prod_{\Sigma_E} \mathbb{G}_m^n$, given by

$$\widehat{T_U} = \{(t_1^\sigma, \dots, t_n^\sigma)_\sigma |\ t^i_\sigma t^{n+1-i}_{\sigma c} = 1\}.$$

The action of $W_{\mathbb{Q}_\ell}$ on $\widehat{T_U}$ is given by $s \cdot (h_\sigma)_\sigma = (h_{s^{-1}\sigma})_\sigma$. A priori E is not Galois over $\mathbb{Q}$. An analogous computation for the maximal (diagonal) torus of $\mathrm{Res}_{E/\mathbb{Q}}\ \mathrm{GL}_n\ T_{GL_{n,E}}$ gives

$${}^LT_{\mathrm{GL}_{n,E}} = \mathbb{G}_m^{n,\Sigma_E} \rtimes W_{\mathbb{Q}_\ell}, \quad s \cdot (t_{i,\sigma}) = (t_{i,s^{-1}\sigma}).$$

and we thus have a natural map ${}^LT_U \mapsto {}^LT_{\mathrm{GL}_{n,E}}$. As π_ℓ is unramified, by [**Bor79**] there is a parameter $r_G : W_{\mathbb{Q}_\ell} \longrightarrow {}^LT$, which we can compose to get

$$r : W_{\mathbb{Q}_\ell} \longrightarrow {}^LT_U,$$

and by the previous map we get and unramified Langlands parameter $r_{\mathrm{GL}_{n,E}} : W_{\mathbb{Q}_\ell} \longrightarrow {}^LT_{\mathrm{GL}_{n,E}}$. Restricting this last parameter to $W_{E_{\lambda'}}$, where $W_{E_{\lambda'}} \longrightarrow W_{\mathbb{Q}_\ell}$ is induced by some $i : E_{\lambda'} \longrightarrow \overline{\mathbb{Q}}_\ell$, we get $W_{E_{\lambda'}} \longrightarrow {}^LT_{\mathrm{GL}_{n,E}} \rtimes W_{E_{\lambda'}}$. Fix an isomorphism $\phi : \overline{\mathbb{Q}}_\ell \longrightarrow \mathbb{C}$, so we can identify complex and ℓ-adic embeddings of E. But the action of $W_{E_{\lambda'}}$ preserves the $\sigma \in \Sigma_E$ over λ', and we can thus project to any such using $\mathrm{pr}_\sigma : {}^LT_{GL_{n,E}} \longrightarrow \mathrm{GL}_n$, so choose the one corresponding to the embedding $W_{E_{\lambda'}} \longrightarrow W_{\overline{\mathbb{Q}}_\ell}$, $\sigma_{\lambda'}$,

$$r_{\lambda'} : W_{E_{\lambda'}} \longrightarrow {}^LT_{\mathrm{GL}_n} =: \widehat{\mathbb{G}_m^n} \times W_{E_{\lambda'}}, w \mapsto r_{\mathrm{GL}_{n,E}}(w) = (h_\sigma)_\sigma \rtimes \pi \mapsto h_{\sigma_{\lambda'}}.$$

Let us show that this is well defined and independant of choices of i and ϕ. Let $i, j : E_{\lambda'} \longrightarrow \overline{\mathbb{Q}_\ell}$ two choices. There exists $s \in W_{\overline{\mathbb{Q}}_\ell}$ such that $s \circ j = i$. These two maps induces two maps $W_{E'_\lambda} \overset{i_*, j_*}{\longrightarrow} W_{\overline{\mathbb{Q}}_\ell}$, such that $j_* = s^{-1} \circ i_* \circ s$.

Moreover using the canonical map $E \longrightarrow E_{\lambda'}$ this induces two embeddings $\sigma^i_{\lambda'}, \sigma^j_{\lambda'} : E \longrightarrow \overline{\mathbb{Q}_\ell}$ above λ' such that $\sigma^i_{\lambda'} = s \circ \sigma^j_{\lambda'}$. So we compute,

$$r_{\mathrm{GL}_{n,E}}(j_\star w) = r_{\mathrm{GL}_{n,E}}(s^{-1} i_* w s) = x \rtimes s^{-1}(h_\sigma)_\sigma \rtimes w (x \rtimes s^{-1})^{-1} = (x (h_{s\sigma})_\sigma s^{-1} w s x^{-1}) \rtimes w,$$

which is mapped under projection to the embedding $\sigma^j_{\lambda'}$ to

$$x_{\sigma^j_{\lambda'}} h_{s\sigma^j_{\lambda'}} x^{-1}_{s^{-1}w^{-1}s\sigma^j_{\lambda'}},$$

but this is commutative, and $w^{-1}s\sigma^j_{\lambda'} = s\sigma^j_{\lambda'}$ as $w \in W_{E_{\lambda'}}$ thus we get

$$x_{\sigma^j_{\lambda'}} x^{-1}_{\sigma^j_{\lambda'}} h_{\sigma^i_{\lambda'}},$$

i.e. $\pi_{\sigma^i_{\lambda'}} \circ r_{\mathrm{GL}_{n,E}} \circ i_* = \pi_{\sigma^j_{\lambda'}} \circ r_{\mathrm{GL}_{n,E}} \circ j_*$ is well defined and independant of the choice of i. Now assume that ϕ, ϕ' are different isomorphisms $\overline{\mathbb{Q}}_\ell \longrightarrow \mathbb{C}$. So for each $i : E_{\lambda'} \longrightarrow \overline{\mathbb{Q}}_\ell$ we get two embeddings of E, namely σ^i_ϕ and $\sigma^i_{\phi'} = s \circ \sigma^i_\phi$, with $s = \phi' \circ \phi \in G_{\overline{\mathbb{Q}}_\ell}$. Thus we are reduced to the previous computation with two different embedding above λ'. Thus $r_{\lambda'} := \pi_{\sigma^i_{\lambda'}} \circ r_{\mathrm{GL}_{n,E}} \circ i_*$ depends only on the choice of $\lambda'|\ \ell$ in E. □

Using the previous proposition, to Π we can for all unramified ℓ associate to Π_ℓ a semi-simple conjugacy class in ${}^LT_U$ and for all $\lambda'|\ \ell$ in E a system of semi-simple conjugacy classes $C_{\lambda'} = r_{\lambda'}(\mathrm{Frob}_{\lambda'})$ in GL_n. We denote $Sat(\Pi_\ell) = (Sat_\lambda(\Pi_\ell))_{\lambda'} =: (C_{\lambda'}|det|^{\frac{1-n}{2}})_{\lambda'}$.

Definition A.2. — Fix an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}_p}$. Let $\rho : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$. We say that ρ is strongly (resp. weakly) essentially associated to Π if for all ℓ (resp. for almost all ℓ), unramified in E and for Π, for all $\lambda'|\ell$, ρ is unramified at λ' and the semi simple class of $\rho(\mathrm{Frob}_{\lambda'})$ and $\iota Sat_{\lambda'}(\Pi_\ell)$ coincides. We say that ρ is ***modular*** if there exists a cuspidal Π as before such that ρ is strongly essentially associated to Π.

Remark A.3. —
1. This is not the natural definition, it would be more adequate to say ***essentially modular***. The reason is that because we want to work at fixed polarisation character, we have ignored the part of the similitude character for Π when looking at $Sat(\Pi_\ell)$. We could do an analogous definition keeping track of the similitude character, but it would be more complicated to describe it, in particular at non split primes when $E/\mathbb{Q}$ is not Galois.
2. It is enough to check the compatibility with the Satake parameter at ℓ totally split in E, in which case the previous association is easier to describe. Indeed, by Chebotarev density theorem the totally split primes in E have density 1, thus ρ is completely determined by the conjugacy class of Frobenius at those primes. Moreover, every $\lambda = \lambda'\lambda'^{,c}|\ell$ is split above F (with λ is a prime of F). Thus $GU_{\overline{\mathbb{Q}}_\ell} \simeq (\prod_{\lambda|\ \ell \text{ in } F} \mathrm{GL}_n) \times \mathbb{G}_m$[23] and the Satake parameter (for GU) associated to Π_ℓ has the form
$$(\mathrm{Diag}(t^\lambda_1, \dots, t^\lambda_n)_\lambda, x).$$
Then $Sat(\Pi_\ell)$ is just the collection
$$((|det|^{\frac{p-1}{2}} \mathrm{Diag}(t^\lambda_1, \dots, t^\lambda_n))_\lambda, (|det|^{\frac{1-p}{2}} \mathrm{Diag}(t^{\lambda,-1}_1, \dots, t^{\lambda,-1}_n))_{\lambda^c}).$$
3. A modular ρ is automatically ε^{n-1}-polarized. Indeed, elements $t \in \widehat{T_U}$ satisfies $t^{-1} = w_0 \cdot t^c$, where w_0 is the longuest Weyl element of GL_n, thus (because of the twist) $\iota Sat_{\lambda'}(\Pi_\ell)^{-1} = \iota Sat_{c\lambda'}(\Pi_\ell)p^{n-1}$. By Chebotarev, this proves the claim.

[23] We should choose a CM type to write this isomorphism properly.

Definition A.4. — We say that a cuspidal automorphic representation π of G or $U(V)$ is sufficiently regular if it is a discrete series at infinity and satisfy property $(\star)$ of **[Lab11]** Section 5.1. This is automatic if the parameter at infinity is regular enough.

Remark A.5. — Because of **[Har90a]** Lemma 3.6.1 and Mirkovic, our *almost generic* points, see Definition 5.4, are sufficiently regular in the previous sense.

Theorem A.6. — *Let π be a cuspidal automorphic representation of $G = GU(V)$ which is cohomological and sufficiently regular. There exists L a Levi subgroup of $\mathrm{Res}_{E/\mathbb{Q}} G_E$, a cuspidal automorphic representation Π_L of $L(\mathbb{A})$ together with an automorphic character χ_L of $L(\mathbb{A})$ such that $\Pi_L \otimes \chi_L^{-1}$ is θ_L-stable and π and Π_L corresponds to each other at all unramified (for π and E) finite places. Moreover each factor of $\Pi_L = \Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_r$ is regular algebraic.*

Proof. — If $F = \mathbb{Q}$ this is **[Mor10]** Corollary 8.5.3, except the last part. But by Shin's appendix **[Gol14]**(24) Theorem 1.1 (iii), $\Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_r$ is moreover regular algebraic as π is. Remark that in this case we don't need π to be sufficiently regular, just cohomological. If $[F : \mathbb{Q}] \geqslant 2$, then we will use **[Lab11]**, thus we need the following lemma. Let us introduce some notation. Let $Z = \{x \in E^\times | N_{E/F}(x) \in \mathbb{Q}^\times\}$ and $Z^1 := \mathrm{Ker}(N_{E/F})_Z \subset Z$. Then Z, Z^1 are tori. Moreover we have a maps

$$0 \longrightarrow Z^1 \longrightarrow Z \times U \longrightarrow G \longrightarrow 0, \tag{10}$$

and the last map is surjective on geometric points. Note that if ℓ is a prime of $\mathbb{Q}$, splitting in E, then the sequence (10) is exact on $\mathbb{Q}_\ell$-points.

Lemma A.7. — *Let π be an irreducible discrete automorphic representation of G such that π_∞ is cohomological for ξ. Then there exists an automorphic discrete representations $\psi \otimes \pi_0$ of $Z(\mathbb{A}) \times U(\mathbb{A})$ such that*

1. *the restriction of $\psi \otimes \pi_0$ to the image of $Z^1(\mathbb{A})$ is trivial;*
2. *$\psi = \psi_\pi$, the restriction of π to Z,*
3. *For all place ℓ of $\mathbb{Q}$, splitting in E, we have $(\pi_\ell)|_{Z(\mathbb{Q}_\ell) \times U(\mathbb{Q}_\ell)} \simeq \psi_\ell \otimes \pi_{0,\ell}$;*
4. *π_0 is cohomological for $\xi|_U$, thus regular;*
5. *$\psi_\infty = \xi_{E_\infty^\times}^{-1}$.*
6. *If ℓ is a prime which is unramified in E, then π_0 is unramified if π is.*

Proof. — This is analogous to the proof of **[HT01]** Theorem VI.2.1. Choosing (g_i) in $G(\mathbb{A})$ such that the $\nu(g_i)$ are representatives of the set $\nu(G(\mathbb{A}))/(\nu(G(\mathbb{Q}))N_{E/F}(Z(\mathbb{A}))$ we get as in **[HT01]**,

$$\begin{array}{ccc} \mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A})) & \longrightarrow & \bigoplus_i \mathcal{A}((Z \times U)(\mathbb{Q})\backslash(Z \times U)(\mathbb{A}))^{Z^1(\mathbb{A})} \\ f & \longmapsto & ((g_i \cdot f)|_{(Z\times U)(\mathbb{A})})_i \end{array}$$

where the $g_i \cdot f$ is denotes the right translate of f. As a consequence we have an isomorphism of $(Z \times U)(\mathbb{A})$-representations

$$\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))|_{(Z\times U)(\mathbb{A})} \simeq \bigoplus_i (\mathcal{A}((Z \times U)(\mathbb{Q})\backslash(Z \times U)(\mathbb{A}))^{Z^1(\mathbb{A})})^{g_i}$$

(24) This more generally applies if E contains an imaginary quadratic field

where the upper script g_i denotes a conjugate action by g_i. This shows that, if π is an automorphic representation of $G(\mathbb{A})$ and if π' is an irreducible subquotient of $\pi|_{(Z\times U)(\mathbb{A})}$, a conjugate of π' by one of the g_i is automorphic and trivial on $Z^1(\mathbb{A})$. Let $\psi\otimes\pi_0$ be an automorphic representation ot $(Z\times U)(\mathbb{A})$ whose conjugate by one of the g_i is isomorphic to a subquotient of $\pi|_{(Z\times U)(\mathbb{A})}$. Moreover, since π is cohomological for ξ, there exists an integer i such that $H^i((\mathrm{Lie}\,G(\mathbb{R}))\otimes_{\mathbb{R}}\mathbb{C},U_\infty,\pi_\infty\otimes\xi')\neq 0$ (for $U_\infty\subset U(\mathbb{R})$ a maximal compact subgroup of $G(\mathbb{R})$). So we can choose $\psi\otimes\pi_0$ such that $H^i((\mathrm{Lie}(Z\times U)(\mathbb{R}))\otimes_{\mathbb{R}}\mathbb{C},U_\infty,\psi_\infty\otimes\pi_{0,\infty}\otimes\xi'|_{(Z\times U)(\mathbb{R})})\neq 0$. This proves that π_0 satisfies property 4 of the statement. The property 1 has already been checked and 2 is clear since $Z(\mathbb{A})$ is in the center of $G(\mathbb{A})$. Property 3 is a direct consequence of the fact that if ℓ is a prime that splits in E, the map $(Z\times U)(\mathbb{Q}_\ell)\to G(\mathbb{Q}_\ell)$ is surjective of kernel $Z^1(\mathbb{Q}_\ell)$. Now assume that ℓ is unramified in E. If π is unramified at ℓ, then π has non zero fixed vector under an hyperspecial subgroup of $G(\mathbb{Q}_\ell)$. As the image of $Z(\mathbb{Q}_\ell)\times U(\mathbb{Q}_\ell)$ has a finite index in $G(\mathbb{Q}_\ell)$, the restriction of π_ℓ to $U(\mathbb{Q}_\ell)$ is isomorphic to a finite direct sum of irreducible representation of $U(\mathbb{Q}_\ell)$ which are conjugated in $G(\mathbb{Q}_\ell)$. As the intersection of an hyperspecial subgroup of $G(\mathbb{Q}_\ell)$ with $U(\mathbb{Q}_\ell)$ is an hyperspecial subgroup of $U(\mathbb{Q}_\ell)$, all irreducible subquotients of $\pi_\ell|_{U(\mathbb{Q}_\ell)}$ have nonzero fixed vectors under some hyperspecial subgroup of $U(\mathbb{Q}_\ell)$. This proves property 6. Property 5 is a direct consequence of the equality $\pi|_{E_\infty^\times}=\xi'_{E_\infty^\times}$ following from the fact that π is cohomological for ξ'. ☐

Thus by [**Lab11**] Cor. 5.3 applied to π_0, which is sufficiently regular thus satisfies property $(\star)$, there is a weak base change i.e. L a standard Levi of $\mathrm{Res}_{E/\mathbb{Q}}\,\mathrm{GL}_{n,E}$ that is θ-stable, and a θ_L-stable discrete automorphic representation of L $\Pi_L=\Pi'_1\otimes\cdots\otimes\Pi'_s$ such that $\Pi'_1\boxplus\Pi'_2\boxplus\cdots\boxplus\Pi'_s$ is a weak base change for π_0. As each Π'_i is discrete, then by the main theorem of [**MW89**] we can write Π'_i as an automorphic induction of $\tau_i\otimes Sp(\ell_i)$ for an integer ℓ_i and τ_i a cuspidal automorphic representation of $\mathrm{GL}_{n_i/\ell_i}(\mathbb{A}_E)$. But the proof of [**Mor10**] 8.5.6 shows that as each Π'_i is θ_{n_i}-stable, each τ_i is θ_{n_i/ℓ_i}-stable. In particular, up to reduce L, choosing $(\Pi_j)_{j=1,\dots,r}$ to be the collection $(\tau_i|\det|^{(\ell_i-2k+1)/2})$ for $i=1,\dots,s$ and $k=1,\dots,\ell_i$, we get that Π_j is cuspidal, and Π_j is θ-stable up to twist (by an automorphic character of L). But by [**Lab11**, Cor. 5.3] again we know that the infinitesimal characters of π_0 and $\Pi'_1\boxplus\cdots\boxplus\Pi'_s=\Pi_1\boxplus\cdots\boxplus\Pi_r$ coincides after base change, in particular the latter representation is regular algebraic. Moreover at unramified places this is compatible with the local base change. ☐

Corollary A.8. — *Let π be a cuspidal automorphic representation of $G=GU(V)$ which is cohomological, sufficiently regular, and unramified outside S, which contains ramified places of E. Then to π is (strongly) essentially associated a unique Galois representation,*

$$\rho^u : G_{E,S}\longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p}),$$

satisfying,

$$(\rho^u)^\vee\simeq(\rho^u)^c\varepsilon^{n-1}.$$

In particular, for all prime $\lambda=v\overline{v}$ of F split in E, not in S, we have that the semi-simple conjugacy class of $\rho^u(\mathrm{Frob}_v)$ is equal to the image of the Satake parameter of $\pi_{0,\lambda}|\det|^{\frac{1-n}{2}}$, seen as a representation of $U(F_\lambda)\overset{\iota_v}{\simeq}GL_{n,E_v}$.

Proof. — The previous proof allow us to reduce to π_0 an automorphic representation of U whose weak base change is $\Pi_1 \boxplus \cdots \boxplus \Pi_r$; θ-stable, each Π_i being automorphic for GL_{n_i}, cuspidal, conjugate self dual up to twist by a character. Thus to π_0 we can associate by **[CH13]** again

$$\rho^u = \rho_{\Pi_1} \oplus \cdots \oplus \rho_{\Pi_r}.$$

As $\Pi_1 \boxplus \cdots \boxplus \Pi_r$ is θ-stable, ρ^u satisfies $(\rho^u)^\vee \simeq (\rho^u)^c \otimes \varepsilon^{n-1}$. On the other hand, we know the compatibility of the association of ρ^u with local Langlands : at all ramified primes $\rho^u_v = LL(\pi_{0,v}|.|^{\frac{1-n}{2}})$, i.e. ρ^u is strongly associated to π. □

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