FAMILIES OF COHERENT PEL AUTOMORPHIC FORMS.

by

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Abstract. — In this article we give a construction of eigenvarieties by geometrically interpolating coherent automorphic sheaves of (PEL) Shimura varieties and their global sections. The new feature is that we particularly study the case of an empty ordinary locus, and thus use a replacement of the canonical subgroup in this situation. We specifically take into account the case of small primes and use this particular construction at a specific endoscopic point to prove new cases of the Bloch-Kato conjecture for characters of an imaginary quadratic field.

Contents

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1. Introduction

Families of automorphic forms have proven to be a great tool in number theory in the last 30 years. Their construction dates back to Hida, [**Hid86**], who first constructed families of *ordinary* modular forms (for the group GL_2). This construction was then improved by Coleman in the 1990's, for *overconvergent, finite slope*, modular forms and rigid spaces over $\mathbb{Q}_p$ (whereas Hida was able to construct his families integrally). One great and yet surprising achievement was the construction soon after by Coleman and Mazur of one rigid space, the Eigencurve, which parametrizes all possible families of overconvergent, finite slope modular forms, i.e. gluing all the families previously constructed.

Before motivating the construction of this spaces, let us say that these constructions have seen many generalisations in different directions. First dealing with level outside p and quaternion algebras by Buzzard [**Buz07**], or for other algebraic groups, like unitary groups, compact at infinity by Chenevier [**Che04**], and to more general groups by [**AS08**] and [**Urb11**] using families of (generalised) modular symbols. More recently, [**AIP15**] have been able to construct families and eigenvarieties for Siegel modular forms using families of automorphic sheaves on the Siegel moduli space. These families of sheaves live in the rigid world, they are Banach sheaves on certain strict neighborhoods of the ordinary locus, that interpolates (in some sense) the classical automorphic vector bundles. This strategy has been extended by [**Kas04**, **Bra13**] in the case of Shimura curves, [**ABI$^+$16**] for Hilbert modular forms, and [**Bra16**] for PEL Shimura varieties for which the ordinary locus in non empty.

These spaces are particularly interesting ; through their local properties (see for example [**BC09**] and [**CH13**] for applications to the Bloch-Kato conjecture, and to constructing Galois representation associated to automorphic representations), but also for their global geometry (see [**LWX17**] and the application to the parity conjecture), which remains completely mysterious in general.

In all cases, the construction goes by constructing huge Banach spaces M together with an action of a (commutative) Hecke algebra $\mathbb{T}$ containing a distinguished compact operator U. With this data, if M is a projective Banach space, we can construct following [**Col97b**] a rigid space $\mathcal{E}$ which parametrises Hecke eigensystems for $\mathbb{T}$ acting on M, for which the eigenvalue for U is non-zero. In [**AS08**] and [**Urb11**], these spaces M are the sections on Shimura varieties of p-adic overconvergent modular symbols, which interpolate the etale cohomology of these varieties. In [**AIP15**] and its generalisations, one first construct varying Banach automorphic sheaves $\omega^{\kappa\dagger}$, where κ is a p- adic weight, and take the sections of these sheaves on strict neighborhoods of the ordinary locus. These spaces interpolate the coherent cohomology, but are constructed on PEL Shimura varieties (one needs the moduli interpretation), and need the non emptyness of the ordinary locus. Indeed, one central tool to construct $\omega^{\kappa\dagger}$ is the theory of the canonical subgroup and its overconvergence (see [**Lub79**, **Far11**] for example). In this article, we mainly remove the ordinariness assumption. Let (G,X) be a PEL Shimura datum[(1)], and p a prime. Our main result is the following

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[(1)] We exclude factors of type D

Theorem 1.1. — *Suppose that G is unramified at p, and let K^p be a level outside p, hyperspecial outside a finite set of primes S. Let I be a Iwahori sugbroup at p and $K = K^pI$. There exists rigid spaces $\mathcal{E}$ and $\mathcal{W}$, called respectively the eigenvariety and the weight space, together with a locally finite map*

$$w : \mathcal{E} \longrightarrow \mathcal{W},$$

and $\mathbb{T} = \mathcal{H}^{Sp} \otimes \mathcal{A}(p) \longrightarrow \mathcal{O}(\mathcal{E})$ such that, for all $\kappa \in \mathcal{W}$, $w^{-1}(\kappa)$ is in bijection with the eigenvalues for the Hecke algebra $\mathbb{T}$ acting on weight κ, overconvergent, locally analytic modular forms for G which are finite slope for some $U \in \mathcal{A}(p)$. Here $\mathcal{A}(p)$ is a (commutative) Hecke algebra at p and $\mathcal{H}^{Sp}$ is the unramified Hecke algebra for G outside Sp. $\mathcal{E}$ and $\mathcal{W}$ are equidimensional of the same dimension. Moreover there is a Zariski dense subset $\mathcal{Z} \subset \mathcal{E}$ such that all $z \in \mathcal{Z}$ coïncide with a classical Hecke eigensystem in the previous identification.

Actually, we can only construct families at unramified primes, but we can weaken a bit the assumptions on G and p, by only constructing deformations in the directions of primes above p which are unramified for G, see remark 2.1. We also have more information on the geometry on $\mathcal{E}$ over $\mathcal{W}$, namely for example there is a covering $(U_i)_i$ of $\mathcal{E}$ such that $w(U_i)$ is an affinoïd open in $\mathcal{W}$, and every irreducible component of U_i surjects via w onto an open of $\mathcal{W}$. Compared to previous constructions of Eigenvarieties (e.g. using modular symbols), the main interest is that we automatically have a broader class of classical points on $\mathcal{E}$, namely those which appear in global sections of coherent automorphic sheaves (with finite slope at p). In particular, automorphic representations which are holomorphic at infinity (but not necessarily discrete series) and finite slope at p can be deformed by our construction (compare with [**Urb11**] 5.5.1). This is particularly interesting to deform the *Arthur points* (an endoscopic point) which we study in the second part of the article in the case of $U(2,1)$, and more generally for endoscopic points (for example those which appear in [**BC09**]), but also for (limit of) discrete series points, for which we would like (for example) to associate Galois representations, as it is done in [**Gol14**].

We now explain how we prove this theorem. A first step in generalising the construction of [**AIP15**] to the case when the ordinary locus is empty is to find a subsitute for the ordinary locus and the canonical subgroup. A good substitute is to consider the μ-ordinary locus (see [**Wed99**], [**Moo04**], and also [**Bij16**]), and the canonical filtration, which exists on it, and overconverges on strict neighborhoods (see [**Her16**]). This strategy has been followed in [**Her19**] for $U(2,1)$ when $p > 2$. Unfortunately, the results of [**Her16**] rely on a stronger hypothesis on p : being big enough (always $p \neq 2$ and for a general unitary group for example the bound can be very large). In this article we choose another strategy to avoid any hypothesis on p, and use (integral) Shimura varieties with higher (Iwahori-like) level at p, constructed by normalisation in [**Lan16a**]. On these Shimura varieties naturally live flags of finite flat subgroups, and if we restrict to strict neighborhoods of the μ-ordinary locus (more precisely what we call the μ-canonical locus, see definition 5.13), these groups behave as the canonical filtration (and actually coincides with it when we know it exists, see Theorem 5.6). In particular, we can follow the construction of [**AIP15**] and [**Her19**] for all p with these groups, and construct automorphic banach sheaves by introducing level at p. All of this rely on the fact that we can find a basis of strict neighborhood $X(deg \geqslant N - \varepsilon)$ where our subgroups have *high degree*, and thus are well behaved.

In the setting where the ordinary locus is non empty, by results of Fargues [**Far11**] we can relate degree and the Hasse invariant. In our situation we also have an Hasse invariant (by [**GN17**]; see also [**Her18**] and Definition 5.11), but we can relate it to the degrees, using [**Her16**], only if p is big enough... Thus we chose another strategy : we have a second basis of strict neighborhoods $X(\mathrm{ha} \leqslant v)$ where the (valuation of) Hasse invariant is small enough (it is invertible on the μ-ordinary locus), and we use these two basis of neighborhoods. Using the degree function, we can control our (call them canonical) subgroups easily, and thus as it was already remarked in [**Bij16**], the action of the Hecke operators. In particular, we can check that we have an operator U which acts as a compact operator on sections of our sheaves over $X(\deg \geqslant N - \varepsilon)$. Unfortunately, we can't prove that the global sections over the opens $X(\deg \geqslant N - \varepsilon)$ of the automorphic Banach sheaves are projective, thus we can't a priori use Coleman- Buzzard's construction. On the other basis $(X(\mathrm{ha} \leqslant v))_{v>0}$, we can't prove even that our expected-to-be compact operator U (which generalise the operator U_p on the modular curve) will stabilise each neighborhood (and thus worse, that it acts compactly on sections on $X(\mathrm{ha} \leqslant v)$), but using that $X(\mathrm{ha} \leqslant v)$ is affinoïd in rigid fiber, we can prove that global section of our automorphic Banach sheaves on $X(\mathrm{ha} \leqslant v)$ are projective. Here to be precise we need to work on both the toroïdal and minimal compactifications of [**Lan16a**], the toroïdal compactification being needed to construct the automorphic sheaves, and the minimal to get the affinoïdness result, together with a result of vanishing of higher cohomology due to Lan, see Appendix A. Thus we need to relate both these sections on the two basis of neighborhoods. Fortunately we can and do in section 9 using complexes computing higher cohomology of our Banach sheaves, the action of the Hecke operators on these complexes, and that we can always intertwine these opens,

$$X(deg \geqslant N - \varepsilon) \supset X(\mathrm{ha} \ \leqslant v) \supset X(deg \geqslant N - \varepsilon') \supset X(\mathrm{ha} \leqslant v') \supset X^{\mu-can},$$

where ε' and v' are chosen small enough. Passing to finite slope parts, and using results of [**Urb11**], we get that U acts as a compact operator on the finite slope part of sections of our Banach automorphic sheaves on any of our strict neighborhoods, and that these spaces are projective (in a specific sense). Thus we can apply Coleman-Buzzard's machinery and get the theorem.

As an application of these results, we can extend the result on the Bloch-Kato conjecture we had in [**Her19**], and prove the following. Let E be a quadratic imaginary number field, and

$$\chi : \mathbb{A}_E^\times / E^\times \longrightarrow \mathbb{C}^\times,$$

which is polarised, meaning that $\chi^\perp := (\chi^c)^{-1} = \chi|.|^{-1}$. Denote by $L(\chi, s)$ its L-function. If p is a prime, denote

$$\chi_p : \mathrm{Gal}(\overline{E}/E) \longrightarrow \overline{\mathbb{Q}_p}^\times,$$

the p-adic Galois character associated to χ, and denote $H^1_f(E, \chi_p)$ the Bloch-Kato-Selmer group of χ_p (see [**BC09**] chapter 5). Then we prove

Theorem 1.2. — *Let p be a prime, unramified in E. If $L(\chi, 0) = 0$ and $\mathrm{ord}_{s=0} L(\chi, s)$ is even, then*

$$H^1_f(E, \chi_p) \neq 0.$$

In particular we remove the hypothesis that $p \neq 2$ when p is inert in E and $p \nmid \mathrm{Cond}(\chi)$ that were in [**Her19**]. Also, a version of the previous theorem is well-known to be due to Rubin ([**Rub91**]) but there it is necessary that $p \neq 2$ (and $p \neq 3$ if $E = \mathbb{Q}(i\sqrt{3})$, which we unfortunately also need to assume...). In particular, we get new cases of the Bloch-Kato conjecture when $p = 2$ is unramified in E !

Of course this result relies heavily, as in [**Her19**], on works of Bellaïche and Chenevier, [**BC04**] and [**BC09**]. As in this last reference, we would like to even construct independent classes as predicted by the Bloch-Kato conjecture, under some assumption on the geometry of the Eigenvariety $\mathcal{E}$ for $U(2,1)$. The idea, as in the proof of the previous theorem, would be to consider a specific *Arthur point* $y \in \mathcal{E}$ (known to exists by results of Rogawski and a calculation of cohomology in [**Her19**]), see Propositon 10.21. As this point exists, we can deduce the previous theorem. Unfortunately, in our situation the motive for this Arthur point appears in degree 0 coherent cohomology with irregular weight (or equivalently when $a \geqslant 2$, not in middle degree Etale cohomology, contrary to the case of [**BC09**]), we are not able to choose a refinement that is *sufficiently far* from the ordinary one for which we can control the ramification at p (we would need it to be *anti-ordinary* as in [**BC04**], but for us the Hodge-Tate weights are in a different order compared to the refinement) but only a slightly non-ordinary one, and thus the geometry of $\mathcal{E}$ at the Arthur point will account for a bigger reducibility locus for the deformation of the Galois representation than expected, and would thus not contribute only to $H^1_f(E, \chi_p)$. We hope to come back on this question soon.

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2. Algebraic groups, Shimura Datum and weight spaces

Let p be a prime and let $\mathcal{D} = (B, \star, V, <,>, \mathcal{O}_B, \Lambda, h)$ be an integral Shimura-PEL-datum. Let G be the associated algebraic group over $\mathbb{Q}$, i.e.,

$$G(R) = \{(g, c(g)) \in GL_B(V \otimes R) \times R^\times \mid < gv, gw >= c(g) < v, w > \forall v, w \in V \otimes R\}.$$

(G, h) defines a Shimura datum. Suppose that the datum is unramified at p (see [**Kot92**] or [**VW13**]). This means that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is isomorphic to a product of matrix algebras over finite extensions of $\mathbb{Q}_p$. We can decompose $B = \prod_{i=1}^r B_i$ as a product of simple algebras and we assume that no factor is of type D (orthogonal), see [**VW13**] Remark 1.1. As p is unramified in $\mathcal{D}$, we can also consider $\mathbb{G}$ a reductive model at p for G (over $\mathbb{Z}_p$).

Every interesting object in this article will decomposed accordingly to the previous decomposition of B, and we can thus make our construction for each B_i. This simple algebras are classified into 2 types (as we excluded case D), the type A and the type C. In case C, the construction we are interested in is already made in [**Bra16**] (which also do many cases of type A, but not all), and we thus assume for now on that B_i is of type A.

As p is unramified for B (and thus B_i) we can further decompose. Let F_i be the center of B_i, and $F_i^+ = (F_i)^{\star=1}$. As we are in case A, $[F_i : F_i^+] = 2$. Write $p = \pi_1 \dots \pi_{s_i}$ the decomposition of p in primes of F_i^+. For $j \in \{1, \dots, s_i\}$, we say that j (or π_j or (B_i, π_j)) is in case AL if π_j splits in F, and in case AU otherwise (compare [**VW13**] Remark 1.3).

Remark 2.1. — Actually we can allow a slightly larger class of Shimura datum than the unramified ones. Suppose that we can write for all i, $B_{i,\mathbb{Q}_p} := B_i \otimes \mathbb{Q}_p = B_{i,1} \times B_{i,2}$ with

$$B_{i,1} = \prod_{j=1}^{s_i} M_{n_{i,j}}(F_{i,j}),$$

where $F_{i,j}/\mathbb{Q}_p$ are finite extensions, and such that there is no factor of type D appearing in $B_{\mathbb{Q}_p}$. For all j denote again $F_{i,j}^+ = (F_{i,j})^{\star=1}$. Let then S_p^{full} be the set of couples (i,j) such that p is not ramified in $F_{i,j}^+$ and does not ramifies in $F_{i,j}$ either. When p is unramified in the datum $\mathcal{D}$, we can take S_p^{full} to be the set of all (i,j) and $B_{i,\mathbb{Q}_p} = B_{i,1}$[2]. In general, for $S_p \subset S_p^{full}$, we will be able to construct S_p-families of automorphic forms for the datum $\mathcal{D}$, i.e. we are able to let the forms vary (only) along the unramified primes of $\mathcal{D}$.

Let T be a maximal torus of $G_1 = \mathrm{Ker}\, c \subset G$ over $\mathbb{Z}_p$ which we assume to be the maximal torus of a Borel defined over $\mathbb{Z}_p$. We can decompose T (over $\mathbb{Z}_p$) according to the previous decomposition,

$$T = \prod_{i=1}^{r} \prod_{j=1}^{s_i} T_{i,j},$$

(remark that if B_i is of type C, we can also decompose according to primes over p).

Definition 2.2. — The full weight space associated to the previous PEL datum is the rigid space over $\mathbb{Q}_p$

$$\mathcal{W}^{full} = \mathrm{Hom}_{cont}(T(\mathbb{Z}_p), \mathbb{G}_m^{rig}),$$

which associate to any Banach $\mathbb{Q}_p$-algebra R the set of continuous characters $\mathrm{Hom}_{cont}(T(\mathbb{Z}_p), R^\times)$. It is represented by the Banach algebra $\mathbb{Z}_p[[T(\mathbb{Z}_p)]]$.

If S_p is a subset of the couples (i,j) (that we see as places over p) and if we denote T_{S_p} the torus over $\mathbb{Z}_p$;

$$T_{S_p} = \prod_{(i,j)\in S_p} T_{i,j},$$

we can define the (S_p-)weight space

$$\mathcal{W}_{S_p} = \mathrm{Hom}_{cont}(T_{S_p}(\mathbb{Z}_p), \mathbb{G}_m^{rig}).$$

It is also represented by the Banach algebra $\mathbb{Z}_p[[T_{S_p}(\mathbb{Z}_p)]]$, and when S_p contains all couples (i,j), we have $\mathcal{W}_{S_p} = \mathcal{W}^{full}$.

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(2) In which case $F_{i,j} = \widehat{(F_i)_{\pi_j}}$

On $\mathcal{W}_{S_p}$ there is a universal character $\kappa^{univ} : T_{S_p}(\mathbb{Z}_p) \longrightarrow \mathbb{Z}_p[[T_{S_p}(\mathbb{Z}_p)]]$. We have the following results,

***Proposition 2.3.** — The space $\mathcal{W}_{S_p}$ is geometrically a finite disjoint union of open balls of dimension the rank of T_{S_p}(3). Moreover there exists an admissible covering by increasing affinoids,*

$$\mathcal{W}_{S_p} = \bigcup_{w>0} \mathcal{W}_{S_p}(w),$$

such that $\kappa^{univ}_{|\mathcal{W}_{S_p}(w)}$ is w-analytic.

Proof. — See **[Urb11]** 3.4.2 and Lemma 3.4.6. See **[AIP15]**, section 2.2 for a possible definition of $\mathcal{W}_{S_p}(w)$. □

We can decompose $\mathcal{W}_{S_p} = \prod_{(i,j)\in S_p} \mathcal{W}_{i,j}$ according to the decomposition of B. In the following we will construct families parametrized by $\mathcal{W}_{S_p}$, as their construction is not more difficult than the case of the full weight space, and following construction can be done on $\mathcal{W}_{S_p}$ when p ramified at some places of $\mathcal{D}$, but not at other places. To my knowledge, this is useful mainly for a trick used by Chenevier (**[Che09]**) to control p-adic properties of families of Galois representations.

3. Classical coherent Automorphic forms

Associated to (G, h) there is a tower of Shimura Varieties over the reflex field E. If we assume that p is unramified in $\mathcal{D}$, these Shimura varieties have good reduction at p when the level at p is hyperspecial (see **[Kot92]**). Suppose this is the case in this section (otherwise all we say here remains true after inverting p, and we will explain how to extends this integrally in section 5). We will describe their integral models as moduli space of Abelian varieties. Let $K^p \subset G(\mathbb{A}^p_{\mathbb{Q},f})$ be sufficiently small level outside p. Denote X_{K^p} the functor,

$$X^{Sph}_{K^p} : S \in Sch/\operatorname{Spec}(\mathcal{O}_{E,p}) \longrightarrow \{(A, i, \lambda, \eta)\}/\sim,$$

that associate the set of quadruples (A, i, λ, η) modulo equivalence where,

- A/S is an abelian scheme
- $\iota : \mathcal{O}_B \longrightarrow \operatorname{End}(A) \otimes \mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$-algebra endomorphism.
- λ is a $\mathbb{Z}^\times_{(p)}$ equivalence class of $\mathcal{O}_B$-linear polarisation of order prime to p which identifies Rosati involution and $\star$ through i.
- η is a K^p-level structure on A (see **[Kot92]** section 5, or **[Lan13]**(4)).

where ι is subject to the determinant condition and the equivalence is by prime to p quasi-isogeny (see also **[VW13]** and for all details **[Lan13]**). As K^p is *sufficently small* $X^{Sph}_{K^p}$ is representible by a quasi-projective smooth scheme.

We choose ν a place of E over p, and denote $\mathcal{O}_{E,\nu}$ the completion of $\mathcal{O}_E$ through ν and denote $X^{Sph} = X^{Sph}_{K^p,\nu}$ the base change to $\mathcal{O}_{E,\nu}$.

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(3) This is the rank of G_1 when p is unramified and $S_p = S_p^{full}$

(4) Recall that such a level structure includes a (class of) isomorphism $\mathbb{Z}/p^N\mathbb{Z} \simeq \mu_{p^N}$ for some N, see **[Lan13]** Definition 1.3.6.1

According to the decomposition of B, we can decompose $A = \prod_{i=1}^r A_i$ (and the other datums) as a product of abelian schemes (with additional structures associated to B_i). Moreover, we can further decompose the associated p-divisible group, writing $\mathcal{O}_{B_i} \otimes \mathbb{Z}_p \simeq \prod_{j=1}^{s_i} M_{n_i}(\mathcal{O}_{F_{i,j}})$, and using Morita-equivalence,

$$A_i[p^\infty] = \prod_{j=1}^{s_i} \mathcal{O}_{F_{i,j}}^{n_i} \otimes_{\mathcal{O}_{F_{i,j}}} A_i[\pi_j^\infty].^{(5)}$$

Moreover for a (i,j) of type AL (i.e. $\pi_j = \pi_j^+ \pi_j^-$ splits in F_i), we can further decompose,

$$A_i[\pi_j^\infty] = H_{i,j} \times H_{i,j}^D,$$

such that λ is given by $(x,y) \mapsto (y,x)$, with $H_{i,j}$ corresponding to $\pi_{i,j}^+$, which we denote, by abuse of notation, $A[\pi_{i,j}^{+,\infty}]$, and ι preserves each factor.

Denote ω the conormal sheaf of A, it is a locally free sheaf on X^{Sph} which decomposes as previously, and for all (i,j) we get $\omega_{i,j} = \omega_{A_i[\pi_j^\infty]}$ a locally free sheaf of rank $\dim A_i[\pi_j^\infty]$. The Shimura datum

$$h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \longrightarrow G_{\mathbb{R}},$$

induces $\mu : \mathbb{G}_{m,\mathbb{C}} \longrightarrow G_{\mathbb{C}}$ (see [**Del71**, Section 3.7]), which is a cocharacter whose conjugacy class is defined over the reflex (number) field E. Let P be the parabolic in G_1 over E associated to the cocharacter $\mu^{(6)}$ and M the Levi of P. T can be seen as a torus in M and fix a Borel B_M of M. For $\kappa \in X^+(T)$ a dominant weight for this choice, there exists a locally free sheaf ω^κ on X^{Sph}. This sheaf can be described this way. Let

$$\mathcal{T}^\times = \mathrm{Isom}_{X^{Sph},\mathcal{O}_B}((\Lambda_1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{X^{Sph}})^\vee, \omega) \simeq \mathrm{Isom}_{X^{Sph},\mathcal{O}_B}(\Lambda_1 \otimes_{\mathbb{Z}_p} \mathcal{O}_{X^{Sph}}, Lie(A/Y)),$$

the space of trivialisations of ω, where Λ_1 is a $\mathcal{O}_B$-invariant $\mathcal{O}_{E,(p)}$-lattice in V_1 (where $V = V_0 \oplus V_1$ under the weight decomposition of μ, see [**VW13**] p10) and denote $\pi : \mathcal{T}^\times \longrightarrow X^{Sph}$. This is a M-torsor.

Definition 3.1. — Let κ be a dominant (in M) algebraic character of T and $\kappa^\vee$ its dual, i.e. $-w_0(\kappa)$ where w_0 is the longest element of the Weyl group of M. We see these characters as characters of B_M, extending them trivially on the unipotent. The coherent automorphic sheaf ω^κ is the locally free sheaf over X^{Sph} defined by,

$$\omega^\kappa = \pi_* \mathcal{O}_{\mathcal{T}^\times}[\kappa^\vee],$$

where $[\kappa^\vee]$ means sections $f : \mathcal{T}^\times \longrightarrow \mathbb{A}^1$ such that $f(gb) = \kappa^\vee(b) f(g)$ for all $g \in \mathcal{T}^\times$ and $b \in B_M$ which acts on the right on $\mathcal{T}^\times$.

Let $X^{Sph,tor}$ be a toroïdal compactification$^{(7)}$ of X^{Sph} (see [**Lan13**]) and D its boundary.

(5) $A[\pi_j^\infty]$ is a slight abuse of notation for the Morita-equivalent p-divisible group associated to the $M_n(\mathcal{O}_{F_{i,j}})$-factor of $A_i[p^\infty]$

(6) i.e. corresponding to the parabolic $P'_{\mathbb{C}} = \{x \in G_{\mathbb{C}} | \lim_{t\to\infty} ad(\mu(t))x \text{ exists}\}$ of $G_{\mathbb{C}}$

(7) A priori the following definition depends on this choice, however by [**Lan13**] Lemma 7.1.1.3, this is independant of the choice of a *toroïdal* compactification, and in most cases we don't even need to specify any compactification, by Koecher's principle, see [**Lan16b**] Theorem 2.3

Definition 3.2. — The space of (respectively cuspidal) modular (or coherent automorphic) forms of weight κ, and level $K^pG(\mathbb{Z}_p)$ is the space,

$$H^0(X^{Sph,tor}, \omega^\kappa), \quad \text{(respectively } H^0(X^{Sph,tor}, \omega^\kappa(-D))\text{)}.$$

Remark 3.3. — The goal of this article is to deform p-adically the previous spaces of automorphic forms. Unfortunately, we can check that in some cases the duality $\kappa \mapsto \kappa' = \kappa^\vee$ does not extend naturally to p-adic weights. This is the case for $U(2,1)_{E/\mathbb{Q}}$ when p is inert in E where $T = \mathcal{O}_{E,p}^\times \times \mathcal{O}_{E,p}^1$. We can see an algebraic weight, dominant for $M \simeq \mathrm{GL}_2 \times \mathrm{GL}_1$, as integers $(k_1 \geqslant k_2, k_3)$. It gives a character of $T(\mathbb{Z}_p)$ via $T(\mathbb{Z}_p) \subset T(K)$, for K a sufficiently large p-adic field (containing E), given by $(x,y) \mapsto \tau(x)^{k_1}\tau(y)^{k_2}\sigma\tau(x)^{k_3}$ and duality (in M) sends $(k_1 \geqslant k_2, k_3)$ to $(-k_2, -k_1, -k_3)$. This does not come from a natural algebraic map on $T(\mathbb{Z}_p)$. The reason is that the embedding $T \longrightarrow M$ is not rational over $\mathbb{Q}_p$. We will overcome this issue by finding a –more or less – natural way to directly interpolate ω^κ without really using a p-adic duality.

4. Local models and Jones Induction result

To construct families of automorphic forms, we will first construct families of automorphic sheaves, i.e. we will construct automorphic sheaves $\omega^{\kappa\dagger}$ for κ not only a dominant algebraic weight but a p-adic one, and these sheaves will interpolate the coherent sheaves ω^κ (actually to be more precise the sheaves $\omega^{\kappa^\vee}$, see remark 3.3). This has been done previously in analogous settings (see [**AIP15**, **AIS14**, **Pil13**, **Bra16**, **Her19**]), and all these works adapt geometrically constructions that were first developed in the case of compact at infinity groups (see [**Buz07**, **Che04**, **Urb11**]) using interpolations of algebraic representations by locally analytic ones. As our sheaves will be modeled on these construction, let us review the theory. It will be useful in analysing classicity questions in section 8.

4.1. Inductions. — Let us fix some notations. We will be interested in representations of a p-adic groups attached to μ. The cocharacter μ gives rise to a parabolic in G, and denote M the Levi subgroup of this parabolic, which is defined over some number field. The group $M_{\mathbb{Q}_p}$ splits over the couples (i,j) introduced before. As explained in the previous section, (i,j) of type (C) are ordinary and thus have been treated in [**Bra16**], thus we focus on type (A). In this cases, $M_{(i,j)}$ is isomorphic to a Levi of the group $\mathrm{Res}_{F_{i,j}^+/\mathbb{Q}_p} U(n_{i,j})_{F_{i,j}/F_{i,j}^+}$.

Denote $\mathcal{T}^+ = \mathcal{T}^+_{(i,j)}$ the set of embeddings of $F_{i,j}^+$ into $\overline{\mathbb{Q}_p}$ and $\mathcal{T}$ the corresponding set for $F_{i,j}$ if (i,j) is understood. $M_{(i,j)}$ is up to extending scalars isomorphic to some $L = \prod_{\tau\in\mathcal{T}^+} \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$ say over K a p-adic field. The integers p_τ, q_τ are determined by μ, the co-character associated to the Shimura Datum (G,h), and satisfy that

$$p_\tau + q_\tau = n_{i,j}, \forall \tau \in \mathcal{T}^+.$$

In particular let K a finite extension of $\mathbb{Q}_p$ such that M is split, and denote T_M its maximal (diagonal) torus. We can assume that we have a fixed isomorphism : $T_K \longrightarrow T_M$ thus we have a map $T(\mathbb{Q}_p) \overset{\iota}{\hookrightarrow} T_M(K)$, which splits over the couples (i,j). For an unramified

$(i,j) \in S_p^{full}$, we can moreover assume $T_{(i,j)}(\mathbb{Z}_p) \overset{\iota}{\hookrightarrow} T_{M,(i,j)}(\mathcal{O}_K)$(8) for some integral model of $M_{(i,j)}$. For now on, we drop the index $(i,j) \in S_p^{full}$ in the notations, thus set $n_{i,j} = n = h = p_\tau + q_\tau$. Still denote by $L = \prod_{\tau \in \mathcal{T}^+} \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$ an integral model over $\mathcal{O}_K$. Let T_M be the maximal (diagonal) torus of L, B the upper Borel, and for each $\kappa \in X^+(T_M) = X^+(T)$, denote the (algebraic, non-normalized) induction,

$$V_\kappa = \{f : L \longrightarrow \mathbb{A}^1 \text{ algebraic } | f(gb) = w_{0,L}\kappa(b^{-1})f(g) \text{ for all } g, b \in L \times B\}.$$

This is a finite dimensional K-vector space endowed with an action of $L(K)$ by $(g.f)(z) = f(g^{-1}z)$.

The algebraic induction is a local model of the automorphic sheaves ω^κ in the sense that etale locally the later is isomorphic to the former. We will now describe another representation that will ***interpolate*** the previous ones and which will be local models of the coherent Banach sheaves constructed later in the paper.

Let $I = I_1$ be the Iwahori subgroup of L, i.e. $I = \mathrm{red}^{-1}(B(\mathcal{O}_K/p))$ where $\mathrm{red} : L(\mathcal{O}_K) \longrightarrow L(\mathcal{O}_K/p)$. Denote more generally I_n the level-n Iwahori, i.e. elements that are upper triangular modulo p^n. We have a Iwahori decomposition $I = B(\mathcal{O}_K) \times N^0$, and we can identify N^0 with

$$(p\mathcal{O}_K)^N \subset \mathbb{A}^N_{an}, \quad N = \sum_{\tau \in \mathcal{T}^+} \frac{p_\tau(p_\tau - 1) + q_\tau(q_\tau - 1)}{2}.$$

For any $\varepsilon \geqslant 0$, we define N^0_ε as the subspace(9),

$$B(N^0, \varepsilon) := \bigcup_{x \in (p\mathcal{O})^N} B(x, \varepsilon) \subset \mathbb{A}^N_{an}$$

and for k a p-adic field, denote $\mathcal{F}^{\varepsilon-an}(N^0, k)$ the function that are restriction to N^0 of analytic functions on N^0_ε. Now we can define the ε-analytic induction. Let $\kappa \in \mathcal{W}(k)$ be ε-analytic, and assume κ extends to an ε-analytic weight κ_K of $T_K(\mathcal{O}_K)$ and write $\kappa_K^\vee : (t \mapsto \kappa_K(w_{0,L}t^{-1}w_{0,L}))$; this preserves ε-analytic characters of $T_K(\mathcal{O}_K)$. Then set

$$V^{\varepsilon-an}_{\kappa_K,k} = \{f : I \longrightarrow k : f(gb) = \kappa_K^\vee(b)f(g) \forall g, b \in I \times B(\mathcal{O}), f_{N^0} \in \mathcal{F}^{\varepsilon-an}(N^0, k)\}.$$

Denote $V^{loc-an}_{\kappa_K,L} = \bigcup_{\varepsilon > 0} V^{\varepsilon-an}_{\kappa_K,L}$ and $V^{an}_{\kappa_K,L} = \bigcap_{\varepsilon \geqslant 0} V^{\varepsilon-an}_{\kappa_K,L}$. This spaces won't be local models of our Banach-automorphic sheaves, but they will have the same finite slope eigenvalues for well chosen κ_K (in particular algebraic ones).

Recall that we have a fixed (i,j). Look at the map $G_{1,(i,j)}(\mathbb{Z}_p) \longrightarrow G_{1,(i,j)}(\mathcal{O}_K)$, and recall that we have $P_{(i,j)}(\mathcal{O}_K) \subset G_{1,(i,j)}(\mathcal{O}_K)$ with Levi $M_{(i,j)}(\mathcal{O}_K) \simeq L(\mathcal{O}_K) = \prod_{\tau \in \mathcal{T}^+} \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}(\mathcal{O}_K)$. Let $P^0_{(p_\tau)_\tau}$ be the preimage of $P_{(i,j)}(\mathcal{O}_K)$ in the Iwahori subgroup of $G_{1,(i,j)}(\mathbb{Z}_p)$. More concretely we can describe it the following way. Choose an ordering, $\{p_\sigma : \sigma \in \mathcal{T}\} = \{p_\tau, q_\tau : \tau \in \mathcal{T}^+\} = \{p_1 \leqslant \dots \leqslant p_{2f}\}$, and let $P_{(p_\sigma)} \subset \mathrm{GL}_n$ be the standard upper parabolic with (ordered) blocks of size $(p_i - p_{i-1})_{i=1,\dots,2f+1}$, where $p_0 = 0$, $p_{2f+1} = n$. Choose an presentation of G_1 (over (i,j)) such that over $\mathbb{Z}_p$ this is the group of matrices with values in $\mathcal{O} = \mathcal{O}_{i,j}$ such that ${}^t\overline{M}JM = J$ with J the antidiagonal

(8)Beware that $\mathbb{Z}_p$-points of $G_{1,(i,j)}$, thus $T_{(i,j)}$ are naturally $\mathcal{O} = \mathcal{O}_{F_{i,j}}$ valued matrices. A priori $\mathcal{O}_K \supset \mathcal{O}_{F_{i,j}}$ (but there is no preferred embedding) but the inclusion is strict.

(9)We set $B(x,r) = \{z \in \mathbb{A}^N_{an} | \ v_p(z - x) \geqslant r\}$ with $v_p(p) = 1$ thus $1 \notin B(0,1) \supset p\mathcal{O}_K$.

matrix with 1's. Then $P^0_{(p_\tau)_\tau} = P^0_{(p_\sigma)}$ is the intersection of $G_{1,(i,j)}(\mathbb{Z}_p)$ with $P_{(p_\sigma)}$. Denote $I^0_{(p_\sigma)}$ the Iwahori subgroup of $P^0_{(p_\sigma)}$ with respect to the standard upper triangular Borel, and $N^0_{(p_\sigma)}$ the opposite unipotent in $I^0_{(p_\sigma)}$. It contains $T(\mathbb{Z}_p)$. For every $\sigma \in \mathcal{T}$, every matrix $M \in P^0_{(p_\sigma)}$ can be written of the form,

$$M = \begin{pmatrix} A_\sigma & B_\sigma \\ 0 & D_\sigma \end{pmatrix}, \quad A_\sigma \in M_{p_\sigma \times p_\sigma}(\mathcal{O}), D_\sigma \in M_{(n-p_\sigma)\times(n-p_\sigma)}(\mathcal{O}).$$

In particular, we get for each $\tau \in \mathcal{T}^+$ a map,

$$\begin{array}{ccc} P^0_{(p_\sigma)} & \longrightarrow & \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}, \\ M & \longmapsto & (\overline{\sigma}(D_{\overline{\sigma}}), \sigma(D_\sigma)) \end{array}, \tag{1}$$

where $\sigma, \overline{\sigma}$ are the two embeddings over $\tau \in \mathcal{T}^+$ such that $p_\sigma = p_\tau = n - p_{\overline{\sigma}}, p_{\overline{\sigma}} = q_\tau$ and D_σ refers to the previous decomposition.

Denote $I_{(p_\sigma)}$ the image of $I^0_{(p_\sigma)}$ via the diagonal morphism $\phi : P^0_{(p_\sigma)} \longrightarrow \prod_\tau \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$ (it is injective on $T(\mathbb{Z}_p)N^0_{(p_\sigma)}$ using that ${}^t\overline{M}JM = J$). Because of the Iwahori decomposition of $I^0_{(p_\sigma)}$, we can write an element of $I_{(p_\sigma)}U(\mathcal{O}_K)$ as n^0tu with n^0 and t in the image of $N^0_{(p_\sigma)}$ and $T(\mathbb{Z}_p)$, and consider, for every $\kappa \in \mathcal{W}(k)$ which is ε-analytic,

$$\begin{aligned} V^{0,\varepsilon-an}_{\kappa,k} = \{f : I_{(p_\sigma)}U(\mathcal{O}_K) \longrightarrow k : f(g\phi(t)u) = \kappa(t)f(g)\,\forall t \in T(\mathbb{Z}_p), u \in U(\mathcal{O}_K), \\ f_{N^0_{(p_\sigma)}} \in \mathcal{F}^{\varepsilon-an}(N^0_{(p_\sigma)}, k)\}. \end{aligned}$$

Everything makes sense as $N^0_{(p_\sigma)}$ can be seen as a subset of N^0 and we can define ε-analytic functions on it (using $B(N^0_{(p_\sigma)}, 1)$ i.e. balls in $\mathbb{A}^N_{an} \supset N^0$ centered on points of $N^0_{(p_\sigma)}$). It is slightly complicated[(10)], but now $V^{0,\varepsilon-an}_{\kappa,k}$ will be local models of our forthcoming Banach-automorphic sheaves. The point is that on $V^{0,\varepsilon-an}_{\kappa,k}$ we really use a (p-adic) weight for $T(\mathbb{Z}_p)$ and not for $T_K(\mathcal{O}_K)$. Now if $\kappa \in X^*(T)$ is an algebraic weight, by scalar extension it corresponds canonically to an algebraic weight of $T_K(\mathcal{O}_K)$ which we see both as a p-adic weight κ of $T(\mathbb{Z}_p)$ and κ_K of $T_K(\mathcal{O}_K)$ (then κ is the restriction of κ_K to $T(\mathbb{Z}_p)$).

Proposition 4.1. — *Let $\varepsilon \leqslant 1$ and $\kappa \in X^*(T)$. The restriction map induces an isomorphism*

$$V^{\varepsilon-an}_{\kappa_K,k} \xrightarrow{\sim} V^{0,\varepsilon-an}_{\kappa,k}.$$

Proof. — First remark that the map ϕ (see equation (1)) sends $T(\mathbb{Z}_p)$ in the torus T_M of L but by $t \mapsto w_{0,L}\iota(t)^{-1}w_{0,L}$ as $D_{\overline{\sigma}} = J_{p_\sigma}{}^t\overline{A_\sigma}^{-1}J_{p_\sigma}$, thus $\kappa_K^\vee \circ \phi(t) = \kappa(t)$. In particular the restriction map is well defined. Moreover as $N^0_1 = B(N^0_{(p_\sigma)}, 1)$ the map is bijective as restriction to N^0_ε (resp. $B(N^0_{(p_\sigma)}, \varepsilon)$) is an isomorphism from $V^{\varepsilon-an}_{\kappa_K,k}$ (resp.

(10) All these constructions are not arbitrary, they come from the analogous geometric situation where $G/\mathbb{Z}_p$ acts on trivialisations of a p-divisible group G, and we want to relate it to trivialisations of the Hodge filtration via $HT_\tau : G^D \longrightarrow \omega_{G,\tau}$, which is modeled by equation (1).

$V^{0-an}_{\kappa,k}$) to $\mathcal{F}^{\varepsilon-an}(N^0,k)$ (resp. $\mathcal{F}^{\varepsilon-an}(N^0_{(p_\sigma)},k)$) (inverse is given by sending f to $n^0tu \mapsto \kappa_K^\vee(t)f(n^0)$). $\square$

4.2. U_p**-operator.** — Define for all $i \leqslant \frac{h}{2}$ an integer,

$$d_i = \begin{pmatrix} p^{-2}I_i & & \\ & p^{-1}I_{h-2i} & \\ & & I_i \end{pmatrix} \in p^{-1}P^0_{(p_\sigma)}(K).$$

We sometimes see d_i in $\mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}$ using the previous embedding. Denote for each $\sigma \in \mathcal{T}$, $a_\sigma = \max(p_\sigma - (h-i), 0), b_\sigma = \max(\min(h-2i, p_\sigma - i), 0)$ and $c_\sigma = \min(i, p_\sigma)$ (thus $a_\sigma + b_\sigma + c_\sigma = p_\sigma$). This is respectively the number of $p^{-2}, p^{-1}, 1$ appearing in $D_{\overline{\sigma}}$ in the previous decomposition for d_i. We can define an operator δ_i on $V^{0,\varepsilon-an}_{\kappa,k}$ by $\delta_i f(j) = f(d_i n d_i^{-1} b)$ where $j = nb$ is the Iwahori decomposition.

Proposition 4.2. — *Let $f \in V^{0,\varepsilon-an}_{\kappa,k}$ that we see as a function in $\mathcal{F}^{\varepsilon-an}(N^0_{(p_\tau)},k)$ of variable $(x^\tau_{k,l}, y^\tau_{m,n})_{1\leqslant l<k\leqslant p_\tau, 1\leqslant n<m\leqslant q_\tau,\tau}$. Then,*

$$\delta_i : \begin{array}{ccc} \mathcal{F}^{\varepsilon-an}(N^0_{(p_\tau)},k) & \longrightarrow & \mathcal{F}^{\varepsilon-an}(N^0_{(p_\tau)},k) \\ f & \longmapsto & ((x^\tau_{k,l}, y^\tau_{m,n}) \mapsto f(p^{v^\tau_{k,l}}x^\tau_{k,l}, p^{w^\tau_{m,n}}y^\tau_{m,n})) \end{array}$$

where, if we denote $\tau = \sigma\overline{\sigma}$ in F, with $p_\tau = p_\sigma$,

$$v^\tau_{k,l} = \begin{cases} 2 & \text{if } k > a_\sigma + b_\sigma \text{ and } l \leqslant a_\sigma \\ 1 & \text{if } (b_\sigma + a_\sigma \geqslant k > a_\sigma \text{ and } l \leqslant a_\sigma) \text{ or } (b_\sigma + a_\sigma \geqslant l > a_\sigma \text{ and } k > a_\sigma + b_\sigma) \\ 0 & \text{otherwise} \end{cases}$$

$$w^\tau_{m,n} = \begin{cases} 2 & \text{if } m > a_{\overline{\sigma}} + b_{\overline{\sigma}} \text{ and } n \leqslant a_{\overline{\sigma}} \\ 1 & \text{if } (b_{\overline{\sigma}} + a_{\overline{\sigma}} \geqslant m > a_{\overline{\sigma}} \text{ and } n \leqslant a_{\overline{\sigma}}) \text{ or } (b_{\overline{\sigma}} + a_{\overline{\sigma}} \geqslant n > a_{\overline{\sigma}} \text{ and } m > a_{\overline{\sigma}} + b_{\overline{\sigma}}) \\ 0 & \text{otherwise} \end{cases}$$

In particular, $\prod_i \delta_i$ is completely continuous.

Remark 4.3. — It is not a mistake that f has "as much variables as entries in N^0" instead of $N^0_{(p_\sigma)}$. The reason is that f is seen as a function (even a locally analytic one) in a neighborhood of the image of $N^0_{(p_\sigma)}(\mathcal{O})$ in the analytic space associated to N^0 (and not to $N^0_{(p_\sigma)}$). Indeed, such f can't be defined on N^0 a priori, except if we know that it is 1-analytic (as the neighborhood of $N^0_{(p_\sigma)}$ of radius $\frac{1}{p}$ in $\mathbb{A}^N = (N^0)^{an}$ contains $N^0 = (p\mathcal{O})^N$.

Proof. — This is a direct calculation on matrices of N^0. $\square$

4.3. Jones's BGG and a fiberwise classicity result. — Let P our previous algebraic group T its torus and B its upper Borel that defines Δ a set of positive roots. Then for every dominant weight $\kappa \in X^+(T)$, Jones's **[Jon11]** proved the exactness of the following sequence,

$$0 \longrightarrow V_{\kappa_K,k} \longrightarrow V^{an}_{\kappa_K,k} \xrightarrow{d} \bigoplus_{\alpha\in\Delta} V^{an}_{\alpha\bullet\kappa_K,k} \tag{2}$$

where d is an explicit map (see for example [**AIP15**] for GSp_{2g} ($P = \mathrm{GL}_g$) and [**Bra16**] for a similar case to ours). Then the following proposition is [**Bra16**] proposition 6.5

Proposition 4.4. — *Write $\kappa = (k_{\sigma,i}) \in X^+(T)$ according to the decomposition $P = \prod_{\tau\in\mathcal{T}^+} \mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau} = \prod_{\sigma\in\mathcal{T}} \mathrm{GL}_{p_\sigma}$ a dominant weight. Set*

$$\nu_i^\sigma = \inf\{k_{\sigma,i} - k_{\sigma,i+1} : i < p_\sigma\}.$$

Then,

$$V_{\kappa,k}^{0,\varepsilon-an,<\underline{\nu}} \subset V_{\kappa,k}.$$

(The same proposition is true with $V_{\kappa,k}^{\varepsilon-an,<\underline{\nu}}$).

Proof. — The first thing to check is that if $f \in V_{\kappa,k}^{0,\varepsilon-an}$ is of non-zero slope, then $f \in V_{\kappa,k}^{an}$ (this reduces to $\varepsilon \leqslant 1$ using Proposition 4.1). But as $\prod_i \delta_i$ is increasing the analytic radius, by proposition 4.2 we get the claim. Now, we can use Jones's BGG result as in [**AIP15**] Proposition 2.5.1, or [**Bra16**] section 6.1, and we get the result. □

Remark 4.5. — The previous calculation is made completely explicitly for $G = (G)U(2,1)$ in [**Her19**].

5. Integral models

5.1. Isogeny Graphs. —

Definition 5.1. — Fix $h \in \mathbb{N}^*$ and $n \in \mathbb{N}^*$, and denote Γ_n^h the subset of $M_{n\times h}(\mathbb{C})$ such that $M = (m_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant h} \in M_{n\times h}(\mathbb{C})$ satisfies,

1. For all (i,j), $m_{i,j} \in \{0,1\}$,
2. For all (i,j), if $m_{i,j} = 1$, then $m_{i-1,j} = 1$ and $m_{i,j-1} = 1$ (when defined).

Let $\underline{\Gamma_n^h} = (\Gamma_n^h, v)$ be the graph whose points are $M \in \Gamma_n^h$, and there is an arrow from $M = (m_{i,j})$ to $M' = (m'_{i,j})$ if

$$\{(i,j) | m_{i,j} \neq m'_{i,j}\} = \{(i_0,j_0)\} \quad \text{and} \quad m_{i_0,j_0} = 0, \quad m'_{i_0,j_0} = 1.$$

When $n = 0$, define Γ_0^h as $\{\star\}$, and the map $\pi_{0,1} : \Gamma_0^h \ni \star \mapsto (0,\dots,0) \in \Gamma_1^h$. When $n \geqslant 2$, we have a natural map,

$$\pi_{n-1,n} : \begin{array}{ccc} \underline{\Gamma_{n-1}^h} & \longrightarrow & \underline{\Gamma_n^h} \\ M = (m_{i,j})_{1\leqslant i\leqslant h, 1\leqslant j\leqslant n-1} & \longmapsto & (m'_{i,j}) \in M_n(\mathbb{C}), m'_{i,j} = m_{i,j} \text{ if } j < n, 0 \text{ otherwise.} \end{array}$$

This map preserves vertices, it is an embedding of graphs.

Remark 5.2. — If $n = 1$, the possible matrices are simply given by

$$M_i = (\underbrace{1,\dots,1}_{i \text{ times}}, 0, \dots, 0).$$

They parametrizes the lattices appearing in a periodic lattice chain inside $\mathrm{GL}_h(\mathbb{Z}_p)$ as in [**RZ96**].

5.2. Some integral models. — Let p be a prime, and let $\mathcal{D}$ be an integral Shimura-PEL-datum as in section 2.

Denote by $\mathcal{P} = \{(i, v) | v \text{ place of } F_i^+\}$ the set of *places* of G, where F_i is the center of B_i. Fix $S_p \subset \{(i, j) | i \in \{1, \dots, r\}, j \in \{1, \dots, s_i\}\}$ a set of unramified places over p as in Remark 2.1. With our assumptions on $\mathcal{D}$, for all $v = (i, j) \in S_p$, v is unramified, and $B \otimes_{F^+} F_v^+$ is split and isomorphic to $M_n(F_v)$. Fix S a finite set of places of $\mathcal{P}$ such that $S \cap S_p = \emptyset$, and S contains all places such that B doesn't split or is ramified.

Fix then a compact K^{S,S_p} outside SS_p such that K_v is maximal hyperspecial for all $v \notin S \cup S_p$.

For all $(i, j) \in S_p$ we can associate an integer $h_{i,j} = \mathrm{ht}_{\mathcal{O}_{i,j}} A[\pi_j]$ in case (AU) and $h_{i,j} = \mathrm{ht}_{\mathcal{O}_{i,j}} A[\pi_j^+]$ in case (AL). These integers are defined for example by looking at the characteristic 0 moduli space as explained in section 3 (or could be read directly on G, and even defined by the integral moduli space of Kottwitz if G is unramified at p). We set $\underline{\Gamma}_n = \prod_{(i,j) \in S_p} \underline{\Gamma}_n^{h_{i,j}}$.

Fix once and for all a compact subgroup $K_S \subset G(F_S)$ and for all $v \in S_p$, consider K_v^{Sph} the maximal hyperspecial compact open subgroup. We will study some covering of the Shimura variety (seen as a scheme over $\mathrm{Spec}(K)$)

$$X^{Sph} = X_{G,K_0}, \quad K_0 = K^{S,S_p} K_S \prod_{v \in S_p} K_v^{Sph}.$$

The Shimura variety associated to X^{Sph} has a good integral model $X_{\mathcal{O}_K}^{Sph}$ over $\mathrm{Spec}(\mathcal{O}_K)$, for $K/\mathbb{Q}_p$ a well chosen finite extension ([**Lan13**] if K_v is hyperspecial for all $v|p$, and [**Lan16a**] if p is unramified in $\mathcal{D}$ for example by normalisation of the hyperspecial level. In general, we fix any integral model $X_{\mathcal{O}_K}^{Sph}$ given by [**Lan16a**]. If K_v is hyperspecial for all $v|p$, $X_{\mathcal{O}_K}^{Sph}$ is smooth).

We will define our base space, and its integral model following [**Lan16a**]. Let for all $v \in S_p$, I_v be a Iwahori subgroup at p of $G(F_v)$. Define first its generic fiber,

$$X = X_{G,K}, K = K^{S,S_p} K_S \prod_{v \in S_p} I_v.$$

This space, over some extension K of $\mathbb{Q}_p$, classifies quintuples $(A, \iota, \lambda, \eta_S, H_\cdot)$ modulo isomorphisms where $(A, \iota, \lambda, \eta_S)$ is a point of X^{Sph}, and $H_\cdot$ is a full flag of $A[v]$, for $v \in S_p$. Explicitly, for every (i, j) as before, $H_\cdot$ induces,

1. In case (AL), a filtration
$$0 \subset H_1 \subset \cdots \subset H_r = A_i[\pi_j^+],$$
by finite flat $\mathcal{O}_{i,j}$-group schemes such that H_k is of rank p^k.
2. In case (AU), a filtration,
$$0 \subset H_1 \subset \cdots \subset H_r = A[\pi_j],$$
by finite flat $\mathcal{O}_{i,j}$-groups schemes such that H_k is of rank p^k and $H_i^\perp = H_{r-i}^{(\sigma)}$.

This Shimura variety with Iwahori level at S_p has a natural integral model over $\mathrm{Spec}(\mathcal{O}_K)$. When all the prime $v|p$ satisfies that K_v is parahoric (this is only a condition outside S_p here), then this is defined by the lattice chain introduced in [**RZ96**]. See for example [**Lan13**]. In general, this can be seen as explained in [**Lan16a**], example 2.4 and 13.12. The abelian scheme A and the subgroups $H_k^{i,j}$ gives rise to isogenies (precisely, we need to use Zarhin's trick, see remark 5.5),

$$A \longrightarrow A_k^{i,j} = A/H_k^{i,j}.$$

In particular we get a map,

$$X \longrightarrow \prod_{\gamma\in\Gamma_1} X_{\mathcal{O}_K}^{Sph},$$

sending $(A,\iota,\lambda,\eta,H.)$ to $(A_k^{i,j},\iota,\eta,\lambda)$ (see remark 5.5). Then the integral model $X_{\mathcal{O}_K}$ is defined as the normalisation of $\prod_{\gamma\in\Gamma_1} X_{\mathcal{O}_K}^{Sph}$ in X. This is a scheme flat over $\mathrm{Spec}(\mathcal{O}_K)$.

The same thing applies to compatible choices of toroïdal compactification[(11)], and we get spaces, flat, proper over $\mathrm{Spec}(\mathcal{O}_K)$ (see [**Lan16a**] Lemma 7.9),

$$X_{\mathcal{O}_K}^{tor} \quad \text{and} \quad X_{\mathcal{O}_K}^{Sph,tor}.$$

Remark 5.3. — In the following, we will be interested mainly by A (as opposed to the collection of all the A_γ) and the subgroups $H_{i,j}^k[p^\ell]$. Thankfully, there is a "universal semi-abelian scheme" (more precisely, a degenerating family) on $\mathfrak{X}^{tor}$ and its covers extending A on X. If p is unramified in the PEL datum and we are at hyperspecial level this is [**Lan13**] Theorem 6.4.1, in general this is [**Lan16a**] Theorem 11.2.

But we will need slightly more, as for a semi-abelian scheme G, $G[p^n]$ need not to be finite flat. Fortunately, we can find an etale covering $\mathcal{U}$ of $X_{\mathcal{O}_K}^{Sph,tor}$ such that G is approximated on each open of this covering by a Mumford 1-motive M, i.e. $G[p^n] = M[p^n]$ (see [**Str10**] section 2.3 (more precisely Proposition 2.3.3.1) and [**Lan16a**] Theorem 11.2). This etale covering is an isomorphism on the boundary (see [**Str10**] section 2.4). In particular, there is a semi-abelian scheme of constant rank $\widetilde{G}$ such that $\widetilde{G}[p^n] \subset G[p^n]$ is finite flat, and such that $\omega_{\widetilde{G}[p^n]} = \omega_{G[p^n]}$. We can thus by pullback find also an etale covering of $X_{\mathcal{O}_K}^{tor}$ on which we have the finite flat group scheme $\widetilde{G}[p^n]$. Thus, the (μ-ordinary) Hasse invariant or the degree function extends on this covering of $X_{\mathcal{O}_K}^{Sph,tor}$, but we can descend them : see in subsection 5.5.

We have similarly for any n, a Shimura variety with Iwahori level p^n, $X_0^{tor}(p^n)$, over $\mathrm{Spec}(K)$, classifying, outside the boundary, $(A,\iota,\lambda,\eta_S,H.)$, with $H.$ a full flag of $A[p^n]$. More precisely, we have for all (i,j),

1. In case (AL), a filtration

$$0 \subset H_1 \subset \cdots \subset H_r = A_i[\pi_j^{+,n}],$$

by finite flat $\mathcal{O}_{i,j}$-group schemes such that H_k is of $\mathcal{O}_{i,j}$-rank p^{nk} with cyclic graded pieces.

(11) In all the this text, we always assume the rational cone decompositions to be smooth and projective without further comment

2. In case (AU), a filtration,
$$0 \subset H_1 \subset \cdots \subset H_r = A[\pi_j^n],$$
by finite flat $\mathcal{O}_{i,j}$-groups schemes such that H_k is of $\mathcal{O}_{i,j}$-rank p^{nk} with cyclic graded pieces such that $H_i^\perp = H_{r-i}^{(\sigma)}$.

Once again, by **[Lan16a]** (here we are in characteristics zero, so this is easier) there is a natural map[(12)] (again, see remark 5.5),
$$X_0(p^n)^{tor} \longrightarrow \prod_{\gamma\in\Gamma_n} X_{\mathcal{O}_K}^{Sph,tor},$$
sending $(A,\iota,\lambda,\eta_S,H.)$ to $(A/(H_{i,j}^k[p^\ell]),\iota,\lambda,\eta_S,)$ away from the boundary.

There is moreover a map
$$X_0(p^n)^{tor} \overset{\pi_{n,n-1}}{\longrightarrow} X_0(p^{n-1})^{tor},$$
given by sending the flag $(H_{i,j}^k[p^\ell] \subset A_i[\pi_j^\ell])_{\ell\leqslant n}$ to the flag $(H_{(i,j)}^k[p^\ell] \subset A_i[\pi_j^\ell])_{\ell\leqslant n-1}$. In particular, the diagram,
$$\begin{tikzcd} X_0(p^n)^{tor} \arrow[r] \arrow[d, "\pi_{n,n-1}"] & \prod_{\gamma\in\Gamma_n} X_{\mathcal{O}_K}^{Sph,tor} \arrow[d, "\Gamma_{n,n-1}"] \\ X_0(p^{n-1})^{tor} \arrow[r] & \prod_{\gamma\in\Gamma_{n-1}} X_{\mathcal{O}_K}^{Sph,tor} \end{tikzcd}$$
is commutative.

Definition 5.4. — Define $X_{0,\mathcal{O}_K}(p^n)^{tor}$ to be the normalisation of $\prod_{\Gamma_n} X_{\mathcal{O}_K}^{Sph,tor}$ in $X_0(p^n)^{tor}$. It is a proper and flat scheme over $\mathrm{Spec}(\mathcal{O}_K)$. By normalisation, the map $\pi_{n,n-1}$ extends as a map,
$$\pi_{n,n-1} : X_{0,\mathcal{O}_K}(p^n)^{tor} \longrightarrow X_{0,\mathcal{O}_K}(p^{n-1})^{tor}.$$

In particular (see also **[FC90a]** Chap I Prop. 2.7), over $X_{0,\mathcal{O}_K}(p^n)^{tor}$ we have by pullback natural isogeny graphs,
$$(A_\gamma)_{\gamma\in\Gamma_n},$$
such that the Kernel of $A_i[\pi_j^\infty] \longrightarrow A_{k,m}^{i,j}$, is a finite flat, at least away from the boundary, $\mathcal{O}_{i,j}$-subgroup of $A_i[\pi_j^n]$ of $\mathcal{O}_{i,j}$-rank p^{km}. We denote it by $H_{k,m}^{i,j}$, or, if (i,j) is understood, $H_k[p^m]$. This makes sense as $H_{k,m} = H_{k,n}[p^m]$. In the rest of the text, we sometimes denotes $G_{(i,j)}$ (or G if (i,j) is understood) the p-divisible group $A_i[\pi_j^\infty]$ (or $A_i[\pi_j^{+,\infty}]$ in case AL).

Remark 5.5. — Actually the construction is slightly more evolved as what than been said, as the abelian varieties $A_\gamma = A/H_{i,j}^k[p^\ell]$ appearing in the isogeny graph might not be principally polarized, thus need not to give a map $X_0(p^n) \longrightarrow X^{Sph}$. But as explained in **[Lan16a]** Proposition 4.12 and Proposition 6.1, we have a map to an auxiliary moduli

(12) Cone decomposition must be chosen appropriately, but we suppose so, without further comment, as it is always possible to refine the choices in order to get the compatibility

problem where $A/H^k_{i,j}[p^\ell]$ is modified to be principally polarized by Zarhin's trick, extends to the integral model $X_{0,\mathcal{O}_K}(p^n)$ (all this works on the compactifications), and we can then deduce the extension of $A/H^k_{i,j}[p^\ell]$ itself.

5.3. Results on the canonical filtration and the Hodge-Tate map. —

Theorem 5.6. — *Let L be a valued extension of $\mathbb{Q}_p$, and G be a truncated level n p-divisible group over $\mathrm{Spec}(\mathcal{O}_L)$ with action of $\mathcal{O}$ and signature (p_τ, q_τ).*

1. *Then there exist at most one sub-$\mathcal{O}$-module H_τ of height np_τ such that,*
$$\deg H_\tau > \sum_{\tau'} \min(np'_\tau, np_\tau) - \frac{1}{2}.$$
We call it the canonical subgroup of height p_τ if it exists.
2. *Moreover, if two sub-$\mathcal{O}$-modules $H_\tau, H_{\tau'}$ of respective height $np_\tau, np_{\tau'}$ as before exists, then,*
$$p_\tau \leqslant p_{\tau'} \quad \textit{if and only if} \quad H_\tau \subset H_{\tau'}.$$
3. *If moreover G is polarized, then H_τ is polarized, i.e.*
$$H_\tau^\perp := (G/H_\tau)^D \hookrightarrow G^D,$$
is identified with the canonical subgroup of height q_τ of G^D.
4. *The group H_τ verifying the previous hypothesis is a step of the Harder-Narasihman filtration of G, it also coincide with the kernel of the Hodge-Tate map,*
$$\alpha_{G,\tau,n-\varepsilon} : G(\mathcal{O}_L) \longrightarrow \omega_{G^D,\tau,n-\varepsilon},$$
where $\varepsilon = \deg_\tau(G/H_\tau)$.
5. *Suppose that H_τ as in 1. exists. The cokernel of the Hodge-Tate map,*
$$\alpha_{G,\tau} \otimes 1 : \ G(\mathcal{O}_L) \otimes \mathcal{O}_L \longrightarrow \omega_{G^D,\tau},$$
is of degree $p^{\frac{\mathrm{Deg}_\tau(G[p]/H_\tau[p])}{p^f-1}}$. In particular, write $\varepsilon_{\tau'} = n\min(p_{\tau'}, p_\tau) - \deg_{\tau'} H_\tau$, then the cokernel of the Hodge-Tate map is killed by $p^{\frac{K_\tau(p_\bullet)+\mathrm{S}_\tau(\varepsilon_\bullet)}{p^f-1}}$, where
$$K_\tau(p_\bullet) = \sum_{i=1}^f p^{f-i}\max(p_{\sigma^i\tau} - p_\tau, 0) \quad \textit{and} \quad \mathrm{S}_\tau(\varepsilon_\bullet) = \sum_{i=1}^f p^{f-i}\varepsilon_{\sigma^i\tau}.$$

Proof. — The first three assertions are Bijakowski's result, **[Bij16]** Proposition 1.24,1.25, 1.30 (see for example **[Her16]** Proposition A.2 for something written for the p^n-torsion). Assertion 4. is proposition 7.8 and 7.7 of **[Her16]** (appliying 7.8 we get a step H'_τ and by 7.7 H_τ and H'_τ coincide with the Kernel of the Hodge-Tate map). It it sufficient to prove 5. for $n = 1$. Remark that our hypothesis for $H_\tau \subset G$ implies the same for $H_\tau[p] \subset G[p]$. Indeed denote $\deg_{\tau'} H_\tau = n\min(p_{\tau'}, p_\tau) - \varepsilon_{\tau'}$, and write the sequence,
$$0 \longrightarrow H_\tau[p] \longrightarrow H_\tau \longrightarrow pH_\tau \longrightarrow 0$$
which is exact in generic fibre, where pH_τ is the schematic adherence of $pH_\tau(\mathcal{O}_C)$ in H_τ. Then $pH_\tau \subset G[p^{n-1}]$, and we have,
$$\deg_{\tau'} H_\tau \leqslant \deg_{\tau'} H_\tau[p] + \deg_{\tau'} pH_\tau,$$

and because pH_τ is of height $(n-1)p_\tau$ and inside $G[p^{n-1}]$, $\deg_{\tau'} pH_\tau \leqslant (n-1)\min(p_\tau, p_{\tau'})$. Thus,

$$\deg_{\tau'} H_\tau[p] \geqslant \min(p_{\tau'}, p_\tau) - \varepsilon'_\tau. \tag{3}$$

Then denote $E = G[p]/H_\tau[p]$. The hypothesis on the degree of H_τ, and thus of $H_\tau[p]$ implies

$$\omega_{H_\tau^D,\tau,\varepsilon} = 0$$

for all $\varepsilon < 1 - \deg_\tau(H_\tau^D)$, in particular, $\varepsilon < 1/2$. Using the same devissage of E as in **[Her16]**, proof of theorem 6.1 implies that

$$\deg \mathrm{Coker}(\alpha_{E,\tau,\varepsilon} \otimes 1) = \deg \mathrm{Coker}(\alpha_{G,\tau,\varepsilon} \otimes 1) = \frac{\mathrm{Deg}_\tau(E)}{p^f - 1}.$$

Using the properties of various $\deg_{\tau'}$, and equation (3) we get the result. □

Remark 5.7. —

1. The principal difference of the previous theorem with **[Her16]** is that we don't a priori have the existence of such groups H_τ. In **[Her16]**, up to taking p big enough to relate the (μ-ordinary) Hasse invariant to the Hodge-Tate map, we have a condition for the existence in terms of the Hasse invariant. In this article, we assure the existence by increasing the level at p in the integral model.
2. The bound given in 5 is interesting in general only when p is big enough compared to (p_τ). If p is small and (p_τ) is too big, then it is more interesting to use the bound given by Fargues (**[Far11]**) which states that (in full generality) the cokernel of the Hodge-Tate map is killed by $p^{\frac{1}{p-1}}$. Note that this is because the definition of the degree which involve taking some determinant.

5.4. Degree function, μ-ordinary locus and Hasse invariants. —

Notations 5.8. — In the subsection 5.2, we fixed a sufficiently big p-adic field K and we have defined, for $* \in \{\emptyset, tor\}$, $X^*, X^{Sph,*}, X_0(p^n)^*$ which are schemes over $\mathrm{Spec}(K)$, together with $X^*_{\mathcal{O}_K}, X^{Sph,*}_{\mathcal{O}_K}, X_{0,\mathcal{O}_K}(p^n)^*$ which are integral models over $\mathrm{Spec}(\mathcal{O}_K)$. In the following we will need to leave the world of schemes, and we thus define $\mathfrak{X}^*, \mathfrak{X}^{Sph,*}, \mathfrak{X}_0(p^n)^*$ as the completion of the previous integral models along their special fibers (we suppress K from the notations). These are thus formal schemes over $\mathrm{Spf}(\mathcal{O}_K)$. We define $\mathcal{X}^*, \mathcal{X}^{Sph,*}, \mathcal{X}_0(p^n)^*$ as the rigid fibers of the integral models. These are rigid analytic spaces over $\mathrm{Spm}(K)$, and when $* = tor$, they coincide with the analytification of the analogous $\mathrm{Spec}(K)$-schemes.

As usual, fix a couple $(i,j) \in S_p$, and suppress it from the notation. Denote by σ a Frobenius (at (i,j)), thus $\mathcal{T}$ is principal homogeneous for the action of σ. For each $\tau \in \mathcal{T}$ is associated an integer p_τ, and thus a subgroup of height np_τ over $\mathfrak{X}_0(p^n)$, $H_\tau \subset G = A[\pi_j^\infty]$ or $A[\pi_j^{+,\infty}]$, which is finite flat and killed by p^n. We can thus, following **[Bij16]**, define for each τ a real-valued function,

$$\deg(H_\tau) : \begin{array}{ccc} \mathcal{X}_0(p^n) & \longrightarrow & [0, n\sum_{\tau'} \min(p_\tau, p_{\tau'})] \\ (A, i, \lambda, \eta, H.) & \longmapsto & \deg(H_\tau) \end{array}$$

and consider $\prod_\tau \deg(H_\tau)$. Then we have the following result of Bijakowski **[Bij16]** Proposition 1.34,

Proposition 5.9. — *The locus where the previous function is maximal in $\mathcal{X}_0(p^n)$, i.e.*

$$\prod_{j=0}^{f-1}\deg(H_{\sigma^j\tau})^{-1}(\{n\sum_{\tau'}\min(p_\tau,p_{\tau'})\}\times\cdots\times\{n\sum_{\tau'}\min(p_{\sigma^{f-1}\tau},p_{\tau'})\}),$$

is included in $\mathcal{X}_0(p^n)^{full-\mu-ord}$, the μ-ordinary locus of $\mathcal{X}_0(p^n)$.

Remark 5.10. — To be precise, as we have fixed the prime $(i,j)\in S_p$, the μ-ordinary locus above, and in the rest of the text (until the conclusion at the end of section 9) if not stated otherwise, is with respect to the prime (i,j).

Definition 5.11. — On $\mathfrak{X}^{Sph}$ we can define a μ-ordinary Hasse invariant ${}^\mu\,\mathrm{Ha}$ (cf. [**Her18**], see also [**GN17**, **KW14**]) which is a section of the sheaf $\det\omega_G^{\otimes(p^f-1)} \pmod p$. This defines a function

$$v({}^\mu\,\mathrm{Ha}):\mathcal{X}^{Sph}\longrightarrow[0,1],$$

which sends a $\mathcal{O}_K$-point to the (truncated by 1) valuation of the μ-ordinary Hasse invariant of the reduction of the corresponding point of $\mathfrak{X}^{Sph}$. In particular we can define by pullback an analogous function on $\mathcal{X}_0(p^n)$, and define

$$\mathcal{X}_0(p^n)^{full-\mu}(v)=v({}^\mu\,\mathrm{Ha})^{-1}([0,v]).$$

Remark 5.12. —

1. In the previous definition, the valuation is normalized by $v(p)=1$, and $\mathcal{X}_0(p^n)^{full-\mu}(0)=\mathcal{X}_0(p^n)^{full-\mu-ord}$, the μ-ordinary locus of $\mathcal{X}_0(p^n)$.
2. Actually by construction we have many maps from $\mathcal{X}_0(p^n)$ to $\mathcal{X}^{Sph}$ (and as much for their integral model), namely one for each $\gamma\in\Gamma_n$. The one we consider above is the canonical one corresponding to the zero-matrix γ (which sends A to A, or $(A_\gamma)_\gamma$ to A_0).

Definition 5.13. — Define $\mathcal{X}_0(p^n)(v)$ as the (union of) connected components of $\mathcal{X}_0(p^n)^{full-\mu}(v))$ which contains a point of maximal degree for the previous function (equivalently, the components where the subfiltration of $H.$ of height np_τ coincides with the canonical filtration in sense of theorem 5.6). We will call $\mathcal{X}_0(p^n)(0)=:\mathcal{X}_0^{\mu-ord-can}(p^n)$ the μ-ordinary-canonical locus of $\mathcal{X}_0(p^n)$. It is an open and closed subset of $\mathcal{X}_0(p^n)^{full-\mu}(0)$ and coincides with the locus of maximal degree of proposition 5.9.

Remark 5.14. — $\mathcal{X}_0(p^n)(v)$ is the analogue of the strict neighborhoods of the ordinary-multiplicative part of the modular curves of level $\Gamma_0(p)$.

Definition 5.15. — For $\underline{\varepsilon}=(\varepsilon_\tau)_\tau$, define the rigid analytic open,

$$\mathcal{X}_0(p^n)((\varepsilon_\tau)_\tau)=\prod_{j=0}^{f-1}\deg(H_{\sigma^j\tau})^{-1}(\prod_{j=0}^{f-1}[n\sum_{\tau'}\min(p_{\sigma^j\tau},p_{\tau'})-\varepsilon_{\sigma^j\tau},n\sum_{\tau'}\min(p_{\sigma^j\tau},p_{\tau'})]).$$

This is a strict neighborhood of the μ-ordinary-canonical locus $\mathcal{X}_0(p^n)(0)$ in $\mathcal{X}_0(p^n)$.

Remark 5.16. — The map $\pi_{n,n-1}$ sends $\mathcal{X}_0(p^n)(\underline{\varepsilon})$ into $\mathcal{X}_0(p^{n-1})(\underline{\varepsilon})$. Indeed, if $\deg H_\tau>n\sum_{\tau'}\min(p_\tau,p_{\tau'})-\varepsilon_\tau$, then because of the generic exact sequence,

$$0\longrightarrow H_\tau[p^{n-1}]\longrightarrow H_\tau\longrightarrow K\longrightarrow 0,$$

and the fact that K is killed by p, thus $\deg K \leqslant \sum \min(p_\tau, p_{\tau'})$ we have that $\deg H_\tau[p^{n-1}] \geqslant (n-1)\sum_{\tau'} \min(p_\tau, p_{\tau'}) - \varepsilon$.

5.5. Extension to the boundary. — We want to extend the previous opens to all $\mathcal{X}_0(p^n)^{tor}$, thus we will need to extends the functions $\deg$ and ${}^\mu\mathrm{Ha}$. The function ${}^\mu\mathrm{Ha}$ can be extended to all $\mathfrak{X}_0(p^n)^{tor}$ (as a section of some $\det(\omega_G)^{\otimes N} \otimes (\mathcal{O}_{\mathfrak{X}_0(p^n)^{tor}}/p)$) by **[Lan17]** Theorem 8.7. For the functions $\deg$, we can also extend it. The group H_{p_τ} is the Kernel of an isogeny of semi-abelian schemes

$$\pi : A \longrightarrow A_\gamma,$$

on $\mathfrak{X}_0(p^n)^{tor}$. Thus, looking at the corresponding map on conormal sheaves we get

$$\pi^* : \omega_{A_\gamma} \longrightarrow \omega_A,$$

and taking determinants gives $\det \pi^* \in H^0(\mathfrak{X}_0(p^n)^{tor}, \det \omega_A \otimes \det \omega_{A_\gamma}^{-1})$. Over $\mathfrak{X}_0(p^n)$, the valuation at every point of $\det \pi^*$, which can be seen as an element of $\mathbb{R}_+$, coincides with the degree of H_{p_τ}. Thus, we have extended the degree map to,

$$\deg(H_\tau) : \mathcal{X}_0(p^n)^{tor} \longrightarrow \mathbb{R}_+.$$

To check that this map is actually bounded by $n\sum_{\tau'} \min(p_\tau, p'_\tau)$ as on the open Shimura variety $\mathcal{X}_0(p^n)$, we can do the following. Let $x \in \mathcal{X}_0(p^n)^{tor}(K)$, and let $\widetilde{G}/\mathcal{O}_K$ be the constant toric rank semi-abelian scheme such that A_x is a quotient by some etale sheaf Y of $\widetilde{G}$ by Mumford's construction. Then by **[FC90b]** Corollary 3.5.11, we have an exact sequence, and taking schematic adherence H_n of $\widetilde{G}[n] \otimes K$ in $A_x[n]$, we have that H_n is isomorphic to $\widetilde{G}[n]$ and whose quotient in $A_x[n]$ is etale. Decompose accordingly $A_{\gamma,x}$ together with the isogeny π (see for example **[FC90b]** Corollary III.7.2), and decompose π_H as $\widetilde{\pi}_H$ along $\widetilde{G}$. Then the degree of π_H is the same as $\widetilde{\pi}_H$ as its quotient is etale. But $\ker \widetilde{\pi}_H$ (which is now finite flat) is of signature smaller than $(n\min(p_\tau, p'_\tau))_{\tau'}$, thus the assertion on its degree.

In particular we can define $\mathcal{X}_0(p^n)^{tor}((\varepsilon_\tau)_\tau)$ and $\mathcal{X}_0(p^n)^{tor}(v)$ as before.

5.6. Two collections of strict neighborhoods. — Recall that in a quasi-compact rigid space X, if $U \subset V \subset X$ are quasi-compact opens, we say that V is a strict neighborhood of U if $(V, X\backslash U)$ is an admissible covering of X. This is in particular the case when U is relatively compact in V over X (**[KL05**, Definition 2.1.1] which is denoted often $U \Subset_X V$), see [**KL05**, Lemma 2.3.3].

The previous opens $\mathcal{X}_0(p^n)^{tor}((\varepsilon_\tau)_\tau)$ and $\mathcal{X}_0(p^n)^{tor}(v)$ both define stricts neighborhoods of the μ-ordinary-canonical locus $\mathcal{X}_0(p^n)^{tor}(0)$. Thus we get the following proposition,

***Proposition 5.17.* —** *For all $v > 0$ there exists $(\varepsilon_\tau)_\tau > 0$ such that,*

$$\mathcal{X}_0(p^n)^{tor}((\varepsilon_\tau)_\tau) \subset \mathcal{X}_0(p^n)^{tor}(v),$$

and for all $(\varepsilon_\tau)_\tau > 0$ there exists $v > 0$ such that,

$$\mathcal{X}_0(p^n)^{tor}(v) \subset \mathcal{X}_0(p^n)^{tor}((\varepsilon_\tau)_\tau).$$

Proof. — Fix $\mathcal{V}$ a strict neighborhood of $\mathcal{X}_0^{can-\mu-\mathrm{ord},tor}(p^n) = \mathcal{X}_0(p^n)^{tor}(0)$. As $(\mathcal{V}, \mathcal{X}_0(p^n)^{tor}\backslash\mathcal{X}_0(p^n)^{tor}(0))$ is an admissible covering, $\mathcal{V}$ contains $\mathcal{X}_0(p^n)^{tor}(v))$ for some $v > 0$. The same applies for $\mathcal{X}_0(p^n)^{tor}((\varepsilon_\tau)_\tau)$. □

Definition 5.18. — We say that $\varepsilon = (\varepsilon_\tau)$ and v are n-compatible, or we say that (ε, v, n) is satisfied, if,

$$\mathcal{X}_0(p^n)^{tor}(v) \subset \mathcal{X}_0(p^n)^{tor}(\varepsilon).$$

Let us explain quickly why we chose this two collections of strict neighborhoods. Classically, we use the stricts neighborhoods $\mathcal{X}(v)$ given by the Hasse invariant to construct eigenvarieties because this is the classical definition of Katz, and as the Hasse invariant is a section of an ample line bundle on the minimal compactification, we get that the ordinary (or μ-ordinary) locus and its strict neighborhoods in the minimal compactification are affinoids. This is a crucial part of the construction described in [**AIP15**]. In many case (see [**Bra16**] or [**Her19**] in the Picard case, and using [**Her16**] in all PEL unramified case when p is big enough), we manage to construct on the opens $\mathcal{X}(v)$ a canonical filtration and control the degree of the subgroups of this filtration explicitly in terms of v. Thus the choice of the strict neighborhoods $\mathcal{X}(v)$ is enough to do all the constructions in these cases. But the classicity results as in [**Buz07**, **Kas06**, **Pil11**, **PS12**, **BPS16**] and in the μ-ordinary case [**Bij16**] relies on the stricts neighborhoods in terms of the degree. So in the unramified PEL case when p is not big enough, it is not clear a priori how to relate the degrees in terms of the Hasse invariant. Nevertheless, the previous proposition will allow us to use the best of both worlds.

We will need to understand the behavior of the strict neighborhoods along finite etale maps.

Lemma 5.19. — *Let $\pi : X \longrightarrow Y$ a finite etale map of quasi-compact rigid spaces. Let $U \subset X$ be a quasi-compact open subset and $V = \pi(U)$ the corresponding open in Y. Let $U_w \subset X$ be a strict neighborhood of U, then $\pi(U_w)$ is a strict neighborhood of V.*

Proof. — This is [**BPS16**] Proposition 4.1.7. □

6. Canonical filtration, Hodge-Tate map and overconvergent modular forms

As before, fix a couple $(i,j) \in S_p$ that will be understood until the rest of this section. Let $v \in v(\mathcal{O}_K)$. In the previous section we defined a rigid open denoted $\mathcal{X}_0(p^n)^{tor}(v)$. We first need an integral (formal) model.

Definition 6.1. — Let $\mathfrak{Bl}_I(v)$ be the blow-up of $\mathfrak{X}_0(p^n)^{tor}$ along the ideal $I = (p^v, {}^\mu\mathrm{Ha})$. Let $\mathfrak{X}_0(p^n)^{tor}(v)^0$ be the open of $\mathfrak{Bl}_I(v)$ where I is generated by ${}^\mu\mathrm{Ha}$, and we denote by $\mathfrak{X}_0(p^n)^{tor,full-\mu}(v)$ the normalisation of $\mathfrak{X}_0(p^n)^{tor}(v)^0$. As this scheme is normal; it has the same connected components than its rigid fiber, and we thus denote $\mathfrak{X}_0(p^n)^{tor}(v)$ the one whose generic fiber is $\mathcal{X}_0(p^n)^{tor}(v)$.

From now on, fix $\underline{\varepsilon} < \frac{1}{2}$. Recall that over $\mathfrak{X}_0(p^n)^{tor}$ we have subgroups $H^m_{p_\tau}$ for $m \leqslant n$ (which are finite flat on $\mathfrak{X}_0(p^n)$ of $\mathcal{O}$-rank mp_τ), but a priori only quasi-finite flat over the boundary.

Proposition 6.2. — *If* $\varepsilon < \frac{1}{2}$, *for every* $v > 0$ *such that*

$$\mathcal{X}_0(p^n)^{tor}(v) \subset \mathcal{X}_0(p^n)^{tor}(\varepsilon),$$

the groups H_τ^m *are finite flat over* $\mathfrak{X}_0(p^n)^{tor}(v)$.

Proof. — Over $\mathfrak{X}_0(p^n)^{tor}$ there is a isogeny

$$A \longrightarrow A_\gamma,$$

of semi-abelian schemes whose Kernel is the group H_τ^m (a priori only quasi-finite flat), and this group is finite flat over $\mathfrak{X}_0(p^n)$. Moreover, by a classical construction, there is an etale covering $\mathfrak{U}^{tor}$ of $\mathfrak{X}_0(p^n)^{tor}$ over which the semi abelian schemes A and A_γ can be approximated by a 1-motive of Mumford M and M_γ (concretely these $\mathfrak{U}_{Sph}^{tor}$ exists for $\mathfrak{X}^{Sph}$ by construction, see e.g. **[Str10]** section 3, and we can moreover assure that $M[p^n]$ and $M_\gamma[p^n]$ are isomorphic to $A[p^n]$ and $A_\gamma[p^n]$, by the arguments of **[Str10]** section 2.3, and take the pull-back via $\mathfrak{X}_0(p^n)^{tor} \longrightarrow \prod_\gamma \mathfrak{X}^{Sph,tor}$). We only need to check that $H_{p_\tau}^m$ is finite flat over $\mathfrak{U}^{tor}(v) := \mathfrak{U}^{tor} \times_{\mathfrak{X}_0(p^n)^{tor}} \mathfrak{X}_0(p^n)^{tor}(v)$. But there is an isogeny over $\mathfrak{U}^{tor}(v)$

$$\pi : A \longrightarrow A_\gamma,$$

such that $\operatorname{Ker}\pi$ is $H_{p_\tau}^m$. Thus for every $\mathcal{O}_K$-point of $\mathfrak{U}^{tor}(v)$, $H_{p_\tau}^m$ is of high degree (in the sense of theorem 5.6). But over $\mathfrak{U}^{tor}(v)$, A and A_γ are associated to Mumford 1-motives M and M_γ by Mumford construction. Thus there exists semi abelian schemes G and G_γ, of constant toric ranks, in the datum of M and M_γ, such that the isogeny π reduces to

$$\pi' : G \longrightarrow G_\gamma.$$

Call $H' = \ker \pi'$. It is finite flat as G and G_γ have constant toric ranks. As $\omega_G \simeq \omega_A$ and $\omega_{G_\gamma} \simeq \omega_{A_\gamma}$, the degree of H' is the same as the one of $H_\tau^m = \operatorname{Ker}\pi$. Thus, away from the boundary, over $\mathfrak{U}(v) := \mathfrak{U}^{tor}(v) \times_{\mathfrak{X}_0(p^n)^{tor}(v)} \mathfrak{X}_0(p^n)(v)$, by unicity in $A[p^n]$ of Theorem 5.6, we have $H' = H_\tau^m$ (it is true for every $\mathcal{O}_K$-point, thus on $\mathfrak{U}(v)$ by normality). In particular, the semi-abelian schemes

$$A/H_\tau^n \quad \text{and} \quad A/H',$$

are isomorphic over $\mathfrak{U}(v)$. But by **[FC90b]** Prop. I.2.7, this implies by normality of $\mathfrak{X}_0(p^n)^{tor}(v)$, and thus of $\mathfrak{U}^{tor}(v)$, that they are isomorphic over $\mathfrak{U}^{tor}(v)$. Thus $H_{p_\tau}^m$ is finite flat. $\square$

6.1. The sheaves $\mathcal{F}$ and integral automorphic sheaves. — We denote

$$\{p_\tau |\ \tau \in \mathcal{T}\} \cup \{0, h\} = \{0 =: p_0 \leqslant p_1 < p_2 < \cdots < p_r \leqslant p_{r+1} := h\}.$$

We define for every $v > 0$ such that $\mathcal{X}_0(p^n)(v) \subset \mathcal{X}_0(p^n)(\underline{\varepsilon})$, a cover of $\mathcal{X}_0(p^n)(v)$. In case (AL) or if $p_r = h$ in case (AU) (in which case $p_1 = 0$ by duality and thus on $\mathfrak{X}$ the universal p-divisible group $A_i[\pi_j^\infty]$ has no multiplicative nor etale part), we set

$$\mathcal{X}_1(p^n)^{tor}(v) := \prod_{k=1}^{r+1} \operatorname{Isom}_{\mathcal{X}_0(p^n)^{tor}(v),pol,\mathcal{O}}(H_{p_k}/H_{p_{k-1}}, \mathcal{O}/p^n\mathcal{O}^{p_k - p_{k-1}}),$$

where[(13)] the condition ***pol*** is trivial in case (AL), and in case (AU) means that we are also given an isomorphism,

$$\nu' : (\mathcal{O}/p^n\mathcal{O})^D \simeq (\mathcal{O}/p^n\mathcal{O})^\sigma,$$

i.e.

$$\mathcal{X}_1(p^n)^{tor}(v) \subset \prod_{k=1}^{r+1} \mathrm{Isom}_{\mathcal{X}_0(p^n)^{tor}(v),\mathcal{O}}(H_{p_k}/H_{p_{k-1}}, \mathcal{O}/p^n\mathcal{O}^{p_k-p_{k-1}}) \times \mathrm{Isom}(\mathcal{O}/p^n\mathcal{O})^D, (\mathcal{O}/p^n\mathcal{O})^\sigma),$$

satisfying the following. There are fixed isomorphisms,

$$\phi_k : (H_{p_k}/H_{p_{k-1}})^D \simeq (H_{p_{r-k+2}}/H_{p_{r-k-1}})^{(\sigma)},$$

induced by $H_{p_k}^\perp \simeq H_{p_{r-k+1}}^{(\sigma)}$, itself induced by the prime-to-p polarisation on $\mathfrak{X}^{tor}$.[(14)]

We require that for all k, the two isomorphisms,

$$\psi_k^D : (\mathcal{O}/p^n\mathcal{O})^{D,p_k-p_{k-1}} \longrightarrow (H_{p_k}/H_{p_{k-1}})^D,$$

and

$$\psi_{r-k+2} : (H_{p_{r-k+2}}/H_{p_{r-k+1}}) \longrightarrow \mathcal{O}/p^n\mathcal{O}^{p_k-p_{k-1}},$$

satisfies $\psi_k^D = \psi_{r-k+2}^{(\sigma),-1}$, after identifying source and target via ν' and ϕ_k.

In this definition we have extended slightly the definitions of the (canonical) subgroups H_k : for $k = 0$ we set $H_0 = \{0\}$ and for $k = r+1$ we set $H_{r+1} = G[p^n]$. If $p_r < h$ in case (AU) (in which case $p_1 > 0$ and on $\mathfrak{X}$ the universal p-divisible group $A_i[\pi_j^\infty]$ has a non-zero multiplicative and etale part), we set

$$\mathcal{X}_1(p^n)^{tor}(v) := \prod_{k=2}^{r} Isom_{\mathcal{X}_0(p^n)^{tor}(v),pol,\mathcal{O}}(H_{p_k}/H_{p_{k-1}}\mathcal{O}/p^n\mathcal{O}^{p_k-p_{k-1}}) \times \mathrm{Isom}_{\mathcal{X}_0(p^n)^{tor}(v)}(H_{p_1}, (\mathcal{O}/p^n\mathcal{O})^{p_1}).$$

Remark 6.3. — 1. The difference in definition in case (AU) is because if $p_r = h$, the group $A_i[\pi_j^m]$ is finite flat and polarised on the all toroïdal compactification, but not if $p_r < h$, because $A_i[\pi_j^m]/H_{p_r}$, which is generically finite etale, is only quasi-finite on the boundary.

2. The point is that $\mathcal{X}_1(p^n)^{tor}(v)$ is a rigid open in (a toroïdal compactification of) the Shimura variety for G of some level (which we could make explicit). Indeed, if we use the definition of **[Lan13]** Definition 1.3.7.4. at our prime (i, j), we see that it amounts to the previous definition : the morphism

$$\nu : \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} \mu_{p^n},$$

there induces a perfect pairing,

$$\mathcal{O}/p^n\mathcal{O} \times (\mathcal{O}/p^n\mathcal{O})^{(\sigma)} \xrightarrow{tr(<,>)} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\nu} \mu_{p^n},$$

(13) We now write H_{p_τ} instead of H_τ^n. Thus $H_{p_k} = H_\tau^n$ if $p_\tau = p_k$ and of $\mathcal{O}$-height np_k.

(14) To be precise, we have on $\mathfrak{X}_0(p^n)^{tor}(\varepsilon)$ a semi-abelian scheme A and $H_{p_\tau}^m$ inside its p-torsion. The group homorphism $\lambda : A \longrightarrow A^{\vee,(\sigma)}$, is a polarisation on $\mathfrak{X}_0(p^n)$, and this polarisation, which identifies $H_{p_k}^\perp$ with $H_{p_{r-k}}^{(\sigma)}$, induces an isomorphism $(H_{p_k}/H_{p_{k-1}})^D \simeq (H_{p_{r-k+1}}/H_{p_{r-k}})^{(\sigma)}$ everywhere. Indeed, it is enough to check it locally and introduce the formal-etale covering $\mathfrak{U}^{tor}(v)$ of subsection 5.5. Over $\mathfrak{U}^{tor}(v)$, the polarization extend as λ an isogeny of 1-motives, thus induces an isogeny λ^{ab} of their abelian parts on which the asserted isomorphism follows from theorem 5.6 and normality of $\mathfrak{U}^{tor}(v)$.

where $\mathrm{tr}(< a, b >) := \mathrm{tr}(a\bar{b})$ is a perfect pairing, and thus induces an isomorphism of $\mathcal{O}$-group schemes

$$\nu' : (\mathcal{O}/p^n\mathcal{O})^D \xrightarrow{\sim} (\mathcal{O}/p^n\mathcal{O})^{(\sigma)}.$$

Let ψ_k and ψ_{r-k+2} be the isomorphism induced by a Level structure in the sense of **[Lan13]**, then let $\Delta_k = p_k - p_{k-1}$,

$$\begin{tikzcd}
(H_{p_k}/H_{p_{k-1}}) \times (H_{p_{r-k+2}}/H_{p_{r-k+1}})^{(\sigma)} \arrow[r, "Weil"] \arrow[d, "\psi_k \times \psi_{r-k+2}^{(\sigma)}"] & \mu_{p^n} \arrow[d, "\nu^{-1}"] \\
(\mathcal{O}/p^n\mathcal{O})^{\Delta_k} \times (\mathcal{O}/p^n\mathcal{O})^{\Delta_k} \arrow[r, "\mathrm{tr}(<,>)_L"] & \mathbb{Z}/p^n\mathbb{Z}
\end{tikzcd}$$

must be commutative, and by compatibility between the polarisations on $A[p^n]$ and L, the two pairings $\mathrm{tr}(<,>)_L \circ (\psi_k \times \psi_{r-k+2}^{(\sigma)})$ and $\mathrm{tr}(<,>)_L \circ (\psi_k \times (\nu' \circ (\psi_k^D)^{-1} \circ \phi_k^{-1}))$ must coincide, thus $\psi_{r-k+2}^{(\sigma)} = \nu' \circ (\psi_k^D)^{-1} \circ \phi_k^{-1}$ by **[Lan13]** Corollary 1.1.4.2.

Definition 6.4. — Let $\mathfrak{X}_1(p^n)^{tor}(v)$ be the normalisation of $\mathfrak{X}_0(p^n)^{tor}(v)$ in $\mathcal{X}_1(p^n)^{tor}(v)$. It is flat, proper and normal over $\mathrm{Spec}(\mathcal{O}_K)$, and moreover we have maps

$$\pi_{n,n-1} : \mathfrak{X}_1(p^n)^{tor}(v) \longrightarrow \mathfrak{X}_1(p^{n-1})^{tor}(v),$$

by normalisation of the map sending (ψ_k) to $(\psi_k[p^{n-1}])$.

Proposition 6.5. — *Assume (ε, v, n). For every τ, there is on $\mathfrak{X}_1(p)^{tor}(v)$ a locally free $\mathcal{O}_{\mathfrak{X}_1(p)^{tor}}$-module of rank p_τ $\mathcal{F}_\tau \subset \omega_{G,\tau}$, (respectively in case (AL) also a sheaf $\mathcal{F}_\tau^\perp \subset \omega_{G^D,\tau}$) containing*

$$p^{\frac{K_\tau(q_\bullet)+S_\tau(\varepsilon_\bullet)}{p^f-1}}\omega_\tau \quad (\textit{respectively}\, p^{\frac{K_\tau(p_\bullet)+S_\tau(\varepsilon_\bullet)}{p^f-1}}\omega_{G^D,\tau}).$$

For all n, it induces by pullback by $\pi_n = \pi_{n,1}$ a sheaf $\mathcal{F}_\tau$ (resp. and $\mathcal{F}_\tau^\perp$) on $\mathfrak{X}_1(p^n)^{tor}(v)$, endowed with a compatible map for all $w_\tau < n - \varepsilon_\tau$, for all $\mathrm{Spec}(R) \subset \mathfrak{X}_1(p^n)^{tor}(v)$,

$$\mathrm{HT}_{\tau,w_\tau} : H_{p_\tau,n}^D(R) \longrightarrow \mathcal{F}_\tau \otimes R_{w_\tau},$$

(resp.

$$\mathrm{HT}_{\tau,w_\tau}^\perp : (H_{p_\tau,n}^\perp)^D(R) = (G[p^n]/H_{p_\tau,n})(R) \longrightarrow \mathcal{F}_\tau^\perp \otimes R_{w_\tau},$$

which induces an isomorphism,

$$H_{p_\tau,n}^D(R) \otimes R_{w_\tau} \longrightarrow \mathcal{F}_\tau \otimes R_{w_\tau},$$

(resp. $\mathrm{HT}_{\tau,w_\tau}^\perp \otimes R_{w_\tau}$ is also an isomorphism).

Proof. — Indeed, we can work locally over $S = \mathrm{Spec}(R)$. We have isomorphisms $(H_{p_k}/H_{p_{k-1}})^D(R) \simeq (\mathcal{O}/p^n\mathcal{O})^{p_k - p_{k-1}}$ but as $H_{p_\tau}^D(R)$ is a $\mathcal{O}/p^n\mathcal{O}$-module killed by p^n and of finite presentation, there exists an isomorphism $H_{p_\tau}^D(R) \simeq (\mathcal{O}/p^n\mathcal{O})^{p_\tau}$. We can thus work as in **[AIP15]** proposition 4.3.1 (see **[Her19]** Proposition 6.1), where the analogs of the proposition are assured by Theorem 5.6, and the construction of $\mathfrak{X}_0(p^n)^{tor}(v)$. $\square$

Proposition 6.6. — *Suppose we are given an isogeny on $\mathfrak{X}_1(p)^{tor}(v)$, $\phi : G' \longrightarrow G$ where G' is a p-divisible group, together with subgroups $H'_{p_\tau} \subset G'[p]$ satisfying the properties of Theorem 5.6. We can thus define $\mathcal{F}'$ for G' similarly. Then the induced map,*

$$\phi^* : \omega_{G'} \longrightarrow \omega_G,$$

sends $\mathcal{F}'$ in $\mathcal{F}$.

Proof. — As the groups in Theorem 5.6 are steps of some Harder-Narasihman filtration, and this filtration is functorial, ϕ induces a map

$$\phi : H'_{p_\tau} \longrightarrow H_{p_\tau}.$$

The rest follows easily (see e.g. [**AIP15**] Proposition 4.4.1). □

6.2. Constructing Banach sheaves. — Out of the universal isomorphisms

$$\psi_k^D : (\mathcal{O}/p^n\mathcal{O})^{p_k - p_{k-1}} \longrightarrow (H_{p_k}/H_{p_{k-1}})^D,$$

on $\mathcal{X}_1(p^n)^{tor}(v)$, we get a (full) flag of $(H_{p_k}/H_{p_{k-1}})^D$, and thus (inductively) of $H_{p_s}^D$ for all s[(15)] by writing for all i, $e_1^k, \dots, e^k_{p_k - p_{k-1}}$ the natural basis of $(\mathcal{O}/p^n\mathcal{O})^{p_k-p_{k-1}}$, and we thus denote x_i^k the corresponding images in $(H_{p_k}/H_{p_{k-1}})^D$ through ψ_k. Choose a lift of this basis,

$$(x_1, \dots, x_{p_s})$$

of $H_{p_s}^D$, and denote Fil_m^ψ the subgroup of $H_{p_s}^D$ generated by $x_1, \dots, x_m$. These subgroups do not depend on the lifts. From now on, fix $v > 0$ such that $\mathcal{X}_0(p^n)^{tor}(v) \subset \mathcal{X}_0(p^n)^{tor}(\underline{\varepsilon})$ (i.e. such that (ε, v, n) is satisfied). In particular, we have the sheaves $\mathcal{F}_\tau$ on $\mathfrak{X}_0(p^n)^{tor}(v)$ and the compatibilities with HT_τ of the proposition 6.5. For simplicity, in case (AL) we call $\mathcal{T}$ the set of embeddings of $\mathcal{O}$ together with their conjugate, and represent its elements by $\tau, \overline{\tau}$. For all $\overline{\tau}$, we mean by $\omega_{\overline{\tau}}$ the sheaf $\omega_{G^D,\tau}$, for $\mathcal{F}_{\overline{\tau}}$ the sheaf $\mathcal{F}_\tau^\perp$ and $\mathrm{HT}_{\overline{\tau}} = \mathrm{HT}_\tau^\perp$. We hope it will not lead to any confusion.

Definition 6.7. — For all τ let $\mathcal{G}r_\tau$ be the Grassmanian parametrizing all complete Flags of $\mathcal{F}_\tau$, and $\mathcal{G}r_\tau^+$ which parametrizes same flags, together with a basis of the graded pieces.

Let $w \leqslant n - \varepsilon_\tau$. For all R' in $R - Adm$, an element $\mathrm{Fil}_\bullet \mathcal{F}_\tau$ of $\mathcal{G}r_\tau(R')$ (respectively $(\mathrm{Fil}_\bullet \mathcal{F}_\tau, w_\bullet)$ of $\mathcal{G}r_\tau^+(R')$) is said to be w-compatible with ψ if $\mathrm{Fil}_\bullet \mathcal{F}_\tau \equiv \mathrm{HT}_\tau(\mathrm{Fil}_\bullet^\psi)$ $\pmod{p^w R'}$ (respectively if moreover $w_i \equiv \psi(x_i) \pmod{p^w R' + \mathrm{Fil}_{i-1} \mathcal{F}_\tau}$). This definition does not depends on the choice of the lifts (x_i).

Of course $\mathrm{Fil}_\bullet \mathcal{F}_\tau$ and $\mathrm{Fil}_\bullet^\psi$ are not always defined for the same index set for $\bullet$. It is understood that we restrict $\bullet$ to the smallest of the two index sets. Let $M = \prod_{\tau \in \mathcal{T}} \mathrm{GL}_{p_\tau} / \mathcal{O}_K$ the linear group with upper triangular Borel B such that $M/B = \mathcal{G}r = \prod_\tau \mathcal{G}r_\tau$ is a flag variety for M. Denote also $U \subset B$ the unipotent radical and $\mathcal{G}r^+ = \prod_\tau \mathcal{G}r_\tau^+ = M/U$. Denote $\mathfrak{M}/\mathrm{Spf}(\mathcal{O}_K)$ the completion along the special fiber, $\mathfrak{T} = \prod_\tau \mathfrak{T}_\tau$ its diagonal torus and for $\omega_\tau > 0$, $\mathcal{T}_{\tau, w_\tau}$ the open which represent the functor $\mathfrak{T}_{\tau,w_\tau}(R) = \ker(\mathfrak{T}_\tau(R) \longrightarrow$

(15) By first taking the full flag of $(H_{p_s}/H_{p_{s-1}})^D$ given previously, and then lifting the one of $(H_{p_s}/H_{p_{s-2}})^D/(H_{p_s}/H_{p_{s-1}})^D \simeq (H_{p_{s-1}}/H_{p_{s-2}})^D$ and so on...

$\mathfrak{T}_\tau(R/p^{w_\tau}R))$. We denote $\mathfrak{T}_w = \prod_\tau \mathfrak{T}_{\tau,w}$ and analogously $\mathfrak{B}_w$ and $\mathfrak{U}_w$ (which acts trivially).

Proposition 6.8. — *For each $\tau \in \mathcal{T}$ and $w_\tau < n - \varepsilon_\tau$, there exists formal schemes,*

$$\mathfrak{IW}^+_{\tau,w_\tau} \xrightarrow{\pi_1} \mathfrak{IW}_{\tau,w_\tau} \xrightarrow{\pi_2} \mathfrak{X}_1(p^n)^{tor}(v),$$

where π_1 is a $\mathfrak{T}_{\tau,w_\tau}$-torsor, and π_2 is affine.

Proof. — We set, following [**AIP15**],

$$\mathfrak{IW}_{\tau,w} : \begin{array}{ccc} R-Adm & \longrightarrow & Sets \\ R' & \longrightarrow & \{w-\text{compatible Fil}_\bullet\ \mathcal{F}_\tau \in \mathcal{G}r_\tau(R')\} \end{array}$$

$$\mathfrak{IW}^+_{\tau,w} : \begin{array}{ccc} R-Adm & \longrightarrow & Sets \\ R' & \longrightarrow & \{w-\text{compatible }(\text{Fil}_\bullet\ \mathcal{F}_\tau, w^\tau_\bullet) \in \mathcal{G}r^+_\tau(R')\} \end{array}$$

These are representable by affine formal schemes (some admissible open in an admissible formal blow-up of the previous Grassmanians). ☐

Fix $w < n - \varepsilon_\tau$ for all τ. We denote by,

$$\mathfrak{IW}^+_w = \prod_\tau \mathfrak{IW}^+_{\tau,w} \xrightarrow{\pi_1} \mathfrak{IW}_w = \prod_\tau \mathfrak{IW}_{\tau,w},$$

and $\mathcal{IW}^+_{\tau,w}, \mathcal{IW}_{\tau,w}, \mathcal{IW}^+_w, \mathcal{IW}_w$ the corresponding generic fibers. Recall that $\mathcal{W}$ is the space of weights, i.e. continuous characters of $T(\mathbb{Z}_p)$. Up to pass to some (i,j), we can assume that

$$T(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}, a_i \in \mathcal{O}^\times, a_i\overline{a_{n+1-i}} = 1 \right\}$$

is (a part of) the maximal torus of $G_1 = \ker c \subset G$. We fix the following embedding

$$\iota : \begin{array}{ccc} T(\mathbb{Z}_p) & \longrightarrow & M(\mathcal{O}_K) = \prod_\tau \mathrm{GL}_{p_\tau}(\mathcal{O}_K) \\ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} & \longmapsto & \begin{pmatrix} \tau(a_{p_\tau})^{-1} & & \\ & \ddots & \\ & & \tau(a_1)^{-1} \end{pmatrix}_\tau \end{array}$$

The order is reversed for the following reason. $T(\mathbb{Z}_p)$ acts naturally on $\mathcal{X}_1(p^n)$ by acting on the left on $(\mathcal{O}/p^n\mathcal{O})^{p_k - p_{k-1}} \longrightarrow (H_{p_k}/H_{p^{k-1}})$ in such a way that $a_1, \dots, a_{p_1}$ acts on the trivialisation of H_{p_1} etc.. But when relating trivialisation of the canonical subgroups by the Hodge-Tate map to the sheaf ω, we need to take a dual, and this reverse the order (it sends a trivialisation M to $J^tM^{-1}J$ with J the matrix of the hermitian form, we see it as antidiagonal with 1's). In particular, the natural action of $T(\mathbb{Z}_p)$ both on $\mathcal{X}_1(p^n)$ and M thus $\mathcal{G}r^+$ now are compatible in the sense of definition 6.7.

We haven't really defined $\mathcal{X}_1(p^n)^{tor}$ (but see remark 6.3), and it will not be useful for us, but in general $\mathcal{X}_1(p^n)^{tor} \longrightarrow \mathcal{X}_0(p)^{tor} = \mathcal{X}^{tor}$ would be a torsor over the group[16]

$$I(p^n) := \left\{ \begin{pmatrix} A_1 & \star & \star & \star \\ & A_2 & \star & \star \\ & & \ddots & \star \\ p\mathcal{O}/p^n & & & A_{r+1} \end{pmatrix} \in G(\mathcal{O}/p^n) : \quad A_i \in I_p(\mathcal{O}/p^n\mathcal{O}) \subset \mathrm{GL}_{p_i - p_{i-1}}(\mathcal{O}/p^n) \right\} \pmod{U_P},$$

where we chose an ordering $\{p_\tau, q_\tau \mid \tau \in \mathcal{T}(= \mathcal{T}_{(i,j)})\} \cup \ \{0, h\} = \{0 \leqslant p_1 < p_2 < \cdots < p_r \leqslant p_{r+1} = h\}$, and $h = h_{(i,j)}$ is the $\mathcal{O}_{(i,j)}$-height of $A_i[\pi_j^\infty]$, I_p denote the standard Iwahori subgroup, and U_P is the standard upper-block-diagonal unipotent associated to $p_1 \leqslant p_2 \cdots \leqslant p_{r+1} = h$ (remember that we fixed a couple (i,j) at this point so here everything is related to the group $G = G_{(i,j)}$ at the place (i,j)). Of course, here we chose a specific pairing so that this parabolic is upper-triangular.

The group $I(p^n)$ does not preserve $\mathcal{X}_1(p^n)^{tor}(v)$: the reason is that the condition on $\mathcal{X}_1(p^n)^{tor}(v)$ to be "close to the μ-ordinary canonical locus" (i.e. that the group of height p_τ have big enough degree) fixes the group of height np_τ to be equal to the canonical (and thus unique) corresponding one. In particular $\mathcal{X}_1(p^n)^{tor}(v) \longrightarrow \mathcal{X}_0(p)^{tor}(v)$ is a torsor over,

$$I^0(p^n) := \mathrm{Im}\left(\left\{ \begin{pmatrix} A_1 & 0 & & 0 \\ & A_2 & & \\ & & \ddots & 0 \\ 0 & & & A_r \end{pmatrix} : \quad \begin{matrix} A_i \in I_p(\mathcal{O}/p^n\mathcal{O}) \subset \mathrm{GL}_{p_i - p_{i-1}}(\mathcal{O}/p^n) \\ {}^tA_i J_{p_i - p_{i-1}} A_{r-i+2}^{(\sigma)} = J_{p_i - p_{i-1}} \end{matrix} \right\} \longrightarrow G(\mathbb{Z}/p^n\mathbb{Z}) \pmod{U_P} \right)$$

with J_s the antidiagonal matrix with entries 1 of size s.

Remark 6.9. — The group $I^0(p^n)$ is related to the group $I^0_{(p_\tau)}$ of section 4

$I^0(p^n)$ is the group of $\mathbb{Z}/p^n$-points of a natural group I^0 defined over $\mathbb{Z}_p$, which contains $T(\mathbb{Z}_p)$[17], and denote $B^0 \supset T$ its upper Borel, and U^0 the unipotent. There is a natural action of I^0 on $\mathfrak{IW}_w^+ \longrightarrow \mathfrak{X}_0(p^n)^{tor}$ with U^0 acting trivially and the action on $\mathfrak{X}_1(p^n)^{tor}$ factors through $I^0(p^n)$. Given a character of $T(\mathbb{Z}_p)$ we see it as a character of $B^0(\mathbb{Z}_p)$ trivial on $U^0(\mathbb{Z}_p)$. A character κ is said to be w-analytic if it extends to a (w-analytic) character of $T(\mathbb{Z}_p)\mathfrak{T}_w$, and we see it as a character of $B^0(\mathbb{Z}_p)\mathfrak{B}_w$ where $U^0(\mathbb{Z}_p)\mathfrak{U}_w$ acts trivially.

Denote by $\pi : \mathfrak{IW}_w^+ \longrightarrow \mathfrak{X}_0(p^n)^{tor}(v)$.

Definition 6.10. — Let κ be a w-analytic character in $\mathcal{W}$. The formal sheaf,

$$\mathfrak{w}_w^{\kappa\dagger} := \pi_* \mathcal{O}_{\mathfrak{IW}_w^+}[\kappa],$$

is a small formal Banach sheaf on $\mathfrak{X}_0(p^n)^{tor}(v)$.

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[16] This is for $p_r = h$, there is an analogous description when $p_r < h$.
[17] Again we have restricted the situation to some (i,j) here...

Here we take κ-variant sections for the action of $B^0(\mathbb{Z}_p)\mathfrak{B}_w$ acting on $\mathfrak{IW}_w^+$ above $\mathfrak{X}_0(p^n)^{tor}(v)$ via the previous explanation. We fix the following notation. If $\kappa \in \mathcal{W}(w) \subset \mathcal{W}$, in particular it is locally analytic, then we denote κ^0 its (analytic) restriction to $\mathfrak{T}_w$.

Proof. — Denote κ^0 the restriction to $\mathfrak{T}_w$ of κ. The map

$$\pi_1 : \mathfrak{IW}_w^+ \longrightarrow \mathfrak{IW}_w,$$

is a torsor over $\mathfrak{T}_w$, thus $(\pi_1)_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa^0]$ is invertible, and

$$\pi_2 : \mathfrak{IW}_w \longrightarrow \mathfrak{X}_1(p^n)^{tor}(v),$$

is affine, thus $(\pi_2 \circ \pi_1)_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa^0]$ is a small formal Banach sheaf. As $\mathfrak{X}_0(p^n)^{tor}(v)$ is quasi-excellent (formally of finite-type over $\mathcal{O}_K$), thus Nagata, the map $\mathfrak{X}_1(p^n)^{tor}(v) \longrightarrow \mathfrak{X}_0(p^n)^{tor}(v)$ is finite, and we can use **[AIP15]** with the action of $B^0(\mathbb{Z}/p^n\mathbb{Z})$. Thus,

$$\mathfrak{w}_w^{\kappa\dagger} = \Big((\pi_2 \circ \pi_1)_*\mathcal{O}_{\mathfrak{IW}_w^+}[\kappa^0](\kappa^{-1})\Big)^{B^0(\mathbb{Z}/p^n\mathbb{Z})},$$

is a small Banach sheaf on $\mathfrak{X}_0(p^n)^{tor}(v)$. $\square$

We would like to descend further to $\mathfrak{X}^{tor}(v)$, i.e. at Iwahori level, unfortunately the map $\mathfrak{X}_0(p^n)^{tor}(v) \longrightarrow \mathfrak{X}^{tor}(v)$ is not finite in general...

Let $\omega_w^{\kappa\dagger}$ be the associated rigid sheaf ([**AIP15**] appendice) on $\mathcal{X}_0(p^n)^{tor}(v) \subset \mathcal{X}_0(p^n)^{tor}(\underline{\varepsilon})$.

6.3. Descent to Iwahori level. — In order to get an action of Hecke operators at p, which are defined at Iwahori level, we will need to descend our construction at this level. Fortunately, this is possible in rigid fiber.

Denote by $U^0(p^n)$ the (diagonal, not just block-diagonal) subgroup,

$$\begin{pmatrix} 1 & \star & & \star \\ & 1 & & \vdots \\ & & \ddots & \star \\ & & & 1 \end{pmatrix} \subset I^0(p^n)$$

Define $\mathcal{X}_0^+(p^n)^{tor}(v)$ as the quotient of $\mathcal{X}_1(p^n)^{tor}(v)$ by $U^0(p^n)$, which doesn't parametrizes trivialisations of the groups $(H_{p_k}/H_{p_{k-1}})^D$ but only full flags of subgroups of this quotients, together with a basis of the graded pieces. Actually we can also define the same way $\mathcal{F}_\tau$ over $\mathcal{X}_0(p^n)^+(v)^{tor}$ (i.e. the sheaves $\mathcal{F}_\tau$ descend). As the action of $U^0(p^n)$ on $\mathcal{X}_1(p^n)^{tor}(v)$ lifts to $\mathcal{IW}_w^+$, denote also by $\mathcal{IW}_w^{0,+}$ the quotient of by $U^0(p^n)$. As $\mathcal{F}_\tau \simeq \omega_\tau$ over $\mathcal{X}_0^+(p^n)^{tor}(v)$ (i.e. after inverting p), we thus have an injection,

$$\mathcal{IW}_w^{0,+} \subset (\mathcal{T}/U)_{\mathcal{X}_0^+(p^n)^{tor}(v)}.$$

Proposition 6.11. — *If $n - \varepsilon_\tau > w > n-1$, then the composite,*

$$\mathcal{IW}_w^{0,+} \hookrightarrow (\mathcal{T}/U)_{\mathcal{X}_0^+(p^n)^{tor}(v)} \longrightarrow (\mathcal{T}/U)_{\mathcal{X}^{tor}(v)},$$

is an open immersion.

Proof. — Denote by $V \subset \mathcal{X}^{tor}(v) = \mathcal{X}_0(p)^{tor}(v)$ the image by π_n of $\mathcal{X}_0^+(p^n)^{tor}(v)$. Up to reducing to a suitable affinoid $U \subset \mathcal{X}_0^+(p^n)^{tor}(v)$, the previous composite map h is given by,

$$h : \coprod_\tau \coprod_{\gamma\in S} M_\tau \begin{pmatrix} 1+p^wB(0,1) & & & \\ p^wB(0,1) & 1+p^wB(0,1) & & \\ & & \ddots & \\ & & & 1+p^wB(0,1) \end{pmatrix} \gamma \longrightarrow \coprod_\tau (\mathrm{GL}_{p_\tau}/U)_{\pi_n(U)},$$

where S is a set of representative of $I^0(p^n)/U^0(p^n)$ in $I(\mathcal{O}) \subset \mathrm{GL}_{p_\tau}(\mathcal{O})$, and M_τ is the matrix relating the basis of $H^D_{p_\tau}$ to the fixed one of ω_τ, which is thus related to the Hodge-Tate map (or equivalently relating a fixed basis of $\mathcal{F}_\tau$ to a fixed one of ω_τ). In particular there exists M_τ^* such that, $M_\tau^* M_\tau = p^c Id_{p_\tau}$ for some c (which we could bound in terms of the Hasse invariant or $\frac{1}{p-1}$, but it is not even necessary). From this, we deduce that $M^* \circ h$ is injective, thus the same thing for h. $\square$

Thus we have a map $g_n : \mathcal{IW}_w^{0,+} \longrightarrow \mathcal{X}^{tor}(v)$, and recall that $\mathcal{X}^{tor}(v)$ is a strict neighborhood of the μ-ordinary canonical locus at Iwahori level. It is not clear that the map,

$$\pi_n : \mathcal{X}_0^+(p^n)^{tor}(v) \longrightarrow \mathcal{X}^{tor}(v),$$

is surjective. But still, having n fixed, $\pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$ describe a basis of strict neighborhoods of $\mathcal{X}^{tor,\mu-can}$ by lemma 5.19.

Definition 6.12. — Assume (ε, n, v) is satisfied. The open $\pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$ is a strict neighborhood of the (μ-canonical) ordinary locus $\mathcal{X}^{tor}(0) = \mathcal{X}^{tor}(0)^{can-\mu-ord}$ included in $\mathcal{X}^{tor}(\underline{\varepsilon})$. On $\pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$, if $w \in]n-1, n-\varepsilon_\tau[$ for all τ, for all κ w-analytic, we define the following sheaf,

$$\omega_w^{\kappa\dagger} = ((g_n)_* \mathcal{O}_{\mathcal{IW}_w^{0,+}})[\kappa].$$

It is called the sheaf of overconvergent, w-analytic modular forms of weight κ. For every $v' > 0$ small enough such that $\mathcal{X}^{tor}(v') \subset \pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$, the module

$$M_w^{\kappa\dagger}(v') = H^0(\mathcal{X}^{tor}(v'), \omega_w^{\kappa\dagger}),$$

(respectively $M^{\kappa\dagger}_{cusp,w}(v') = H^0(\mathcal{X}^{tor}(v'), \omega_w^{\kappa\dagger}(-D))$) is called the module of v'-overconvergent, w-analytic (respectively cuspidal) modular forms of weight κ.

Remark 6.13. — In the previous compatibilities, if (ε, n, v) is satisfied, (ε, n, v') is for all $v' < v$. Also, because of the compatibility between w and n, n is uniquely defined (and is thus suppressed form the notation of $\omega_w^{\kappa\dagger}$). Thus, we can choose v arbitrarily close to 0 in the previous definition. Also, for every w and κ, there exists n_0 such that for all $n \geqslant n_0$, there is $w' > w$, and κ is w'-analytic with $n-1 < w' < n - \varepsilon_\tau$ for all τ. In particular, there exists constants v_0, w_0 such that $M_w^{\kappa\dagger}(v)$ is defined for all $v < v_0$ and $w > w_0$ such that $w \in]n-1, n-\varepsilon[$ (for some integer n large enough).

Suppose $n' - \varepsilon_\tau > w' > w$ with $w \in]n-1, n-\varepsilon[$ and $n \leqslant n'$. As a flag with graded basis which is (n', w')-compatible is also (n, w)-compatible, there is an injective map,

$$\mathcal{IW}^+_{n',w'} \hookrightarrow \mathcal{IW}^+_{n,w} \times_{\mathcal{X}_1(p^n)^{tor}(v)} \mathcal{X}_1(p^{n'})^{tor}(v).$$

In particular, we have a natural map, for every w'-analytic κ,

$$\omega_w^{\kappa\dagger} \hookrightarrow \omega_{w'}^{\kappa\dagger},$$

over $\pi_{n'}(\mathcal{X}_0^+(p^{n'})^{tor}(v)) \subset \pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$.

Definition 6.14. — For $w > 0$, the module,

$$M^{\kappa\dagger} = \varinjlim_{v\to 0, w\to\infty} M_w^{\kappa\dagger}(v) \quad (\text{respectively } M_{cusp}^{\kappa\dagger} = \varinjlim_{v\to 0, w\to\infty} M_{cusp,w}^{\kappa\dagger}(v))$$

is the module of overconvergent locally analytic (respectively cuspidal) modular forms of weight κ.

Remark 6.15. — In the previous definition, it is understood that the limit is taken on v, w such that $w \in]n-1, n-\varepsilon[$ for some $n = n_w$, (ε, n, v_0) is satisfied for some v_0 and $\mathcal{X}^{tor}(v) \subset \pi_n(\mathcal{X}_0^+(p^n)^{tor}(v_0)) \subset \mathcal{X}^{tor}(\varepsilon)$. Thus in particular (ε, n, v_0) is satisfied and $v \leqslant v_0$.

Let κ be a classical weight of $\mathcal{W}$. This means that if M denote the Levi associated to μ as in Section 3, we can embed $T(\mathbb{Z}_p)$ in T_M the torus of M and κ is the composition of this embedding with a dominant algebraic character of T_M. If we write $M = \prod_{\tau\in\mathcal{T}} \mathrm{GL}_\tau$, and we choose B the upper Borel, then a dominant algebraic character of T_M can be seen as integers $(k_1 \geqslant \cdots \geqslant k_{p_\tau})_\tau$. We then define the associated character of $T(\mathbb{Z}_p)$ as $-w_{0,M}(\kappa) \circ \iota$ with the embedding ι given in (6.2).

Proposition 6.16. — *Suppose that $\kappa \in \mathcal{W}$ is a dominant algebraic character, and choose w, and n, v and any v_0 such that $w \in]n-1, n-\varepsilon[$, (ε, n, v_0) is satisfied, and $\mathcal{X}^{tor}(v) \subset \pi_n(\mathcal{X}_0^+(p^n)^{tor}(v_0))$. Then we have the following inclusion as sheaves over $\mathcal{X}^{tor}(v)$,*

$$\omega^\kappa \subset \omega_w^{\kappa\dagger}.$$

Proof. — Indeed, sections of ω^κ are by definition section of $\mathcal{O}_{\mathcal{T}^\times}$ which are are $\kappa^\vee = -w_0(\kappa)$-equivariant for the action of the Borel $B \subset \prod_\tau \mathrm{GL}_{p_\tau}$ (with U acting trivially), thus we have by Proposition 6.11 a restriction map $\pi_{\mathcal{T}*}\mathcal{O}_{\mathcal{T}^\times/U} \longrightarrow (g_n)_*\mathcal{O}_{\mathcal{IW}_w^{0,+}}$ over $\mathfrak{X}^{tor}(v)$ which is injective by analytic continuation. But because of the previous definition of κ as a character of $T(\mathbb{Z}_p)$ and the construction both of ω^κ (as $-w_0(\kappa)$ variant function for T_M) and $\omega_w^{\kappa\dagger}$ (as κ-variant functions on $T(\mathbb{Z}_p)$) the previous restriction map factors as an injective map $\omega^\kappa \longrightarrow \omega_w^{\kappa\dagger}$. □

Remark 6.17. — In the previous definition, it can seem a bit arbitrary the use of the map ι from $T(\mathbb{Z}_p)$ to T_M, but it is the natural one from the point of view of the Hodge-Tate map (which relate a trivialisation of G^D, ordered by the canonical filtration, and a trivialisation of ω_G) : it is what assures the compatibility between the action of $T(\mathbb{Z}_p)$ on trivialisations of $G[p^n]$-points, and $\mathfrak{T}_w$ on $\mathcal{IW}_w^+$.[18]. In particular, we have that in case AL – i.e. when primes above p splits in F/F^+– so that we can identify $T(\mathbb{Z}_p)$ with (product of) $\mathcal{O}_K^n$ for some p-adic field K (choosing a CM type above p in F), dominant algebraic weights of

[18] Thus there is a mistake in **[Her19]** in the way we chose the embedding of $T(\mathbb{Z}_p)$ in T_M in section 7.1 which implies that classical sheaves are not associated to p-adic characters $(k_1 \geqslant k_2, k_3)$ but to $(-k_1, -k_2, k_3) \in \mathcal{W}$ with $k_1 \geqslant k_2$ which is unfortunate... The embedding should be given by ι.

T_M corresponds to ***dominant*** integers $(\ell_1^\tau \geqslant \cdots \geqslant \ell_n^\tau)_\tau$ for all embeddings $\tau : K \longrightarrow \overline{\mathbb{Q}_p}$ for sufficiently generic classical points on the Eigenvariety.

6.4. Some complexes. — For compatibilities reasons with Hecke operators and to control the structure on the previous modules, we will need to define complexes overconvergent sections. Recall that on $\mathcal{X}^{tor} = \mathcal{X}_0(p)^{tor}$, our rigid toroïdal Shimura variety with fixed Iwahori level, we have defined two basis of strict neighborhoods of $\mathcal{X}^{tor}(0)$ (the *canonical* μ-ordinary locus whose points have maximal degree), one given by

$$\mathcal{X}^{tor}(\underline{\varepsilon}),$$

for $\underline{\varepsilon} = (\varepsilon_\tau)_\tau$ (of points with degrees bigger than the maximal one minus ε_τ), and

$$\mathcal{X}^{tor}(v),$$

for $v \in (0,1]$ (describing the connected components containing $\mathcal{X}^{tor}(0)$ of points with Hasse invariant of valuation smaller than v). Because we will need to let the neighborhood described in terms of the degree vary, we from now on call $\underline{\varepsilon}_0$ a number fixed to be able to define the sheaves $\omega_w^{\kappa\dagger}$, and we will always consider small enough opens $\mathcal{X}(v)$ and $\mathcal{X}(\underline{\varepsilon})$ so that there exists the sheaves $\omega_w^{\kappa\dagger}$ on them. In particular, once w is fixed this just implies that v or ε are small enough (depending on w).

Ultimately we are interested by (finite slope) overconvergent cuspidal modular forms, that is, (finite slope) elements of

$$\varinjlim_{v\longrightarrow 0, w\longrightarrow\infty} H^0(\mathcal{X}^{tor}(v), \omega_w^{\kappa\dagger}(-D)) = \varinjlim_{\underline{\varepsilon}\longrightarrow 0, w\longrightarrow\infty} H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_w^{\kappa\dagger}(-D)).$$

We temporarily introduce the following complexes,

Definition 6.18. — Let $w > 0, \mathcal{U} = \mathrm{Spm}(A) \subset \mathcal{W}$ an affinoid such that $\kappa_{\mathcal{U}}$ is w-analytic, and define for $v, \underline{\varepsilon}$ small enough [(19)],

$$C_{cusp}(v, w, \kappa_{\mathcal{U}}) = R\Gamma(\mathcal{X}^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}\dagger}(-D)),$$

and

$$C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{U}}) = R\Gamma(\mathcal{X}^{tor}(\underline{\varepsilon}) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}\dagger}(-D)).$$

We can analogously define the non-cuspidal versions of these complexes.

We also define

$$H^i_{cusp,\dagger}(\kappa_{\mathcal{U}}) = \lim_{v\longrightarrow 0, w\longrightarrow\infty} H^i(\mathcal{X}^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}\dagger}(-D)),$$

and

$$H^{i'}_{cusp,\dagger}(\kappa_{\mathcal{U}}) = \lim_{\underline{\varepsilon}\longrightarrow 0, w\longrightarrow\infty} H^i(\mathcal{X}^{tor}(\varepsilon) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}\dagger}(-D)).$$

In particular $H^0_{cusp,\dagger}(\kappa) = H^{0'}_{cusp,\dagger}(\kappa)$ is simply the space of overconvergent locally analytic cuspidal modular forms of weight κ, and we will see that the higher cohomology groups vanishes (their finite slope part at least).

Proposition 6.19. — *The previous complexes are represented by bounded complexes of projective $A[1/p]$-modules (i.e. perfect complexes in the sense of* **[Urb11]***).*

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(19) such that the sheaves $\omega_w^{\kappa_{\mathcal{U}}\dagger}$ is defined on $\mathcal{X}^{tor}(v) \times \mathcal{U}$ resp. $\mathcal{X}^{tor}(\underline{\varepsilon}) \times \mathcal{U}$.

Proof. — This is the same proof as [**Pil18**] Proposition 12.8.2.1. We have maps,

$$\mathcal{IW}_w^+ \times_{\mathcal{X}(v)} \mathcal{U} \xrightarrow{\pi_1} \mathcal{IW}_w \times_{\mathcal{X}(v)} \mathcal{U} \longrightarrow \mathcal{X}_1(p^n)^{tor}(v) \times \mathcal{U},$$

and sheaves $\mathcal{L}^{\kappa_{\mathcal{U}}} = (\pi_{1*}\mathcal{O}_{\mathcal{IW}_w^+})[\kappa_{\mathcal{U}}^0]$ (for the action of $\mathcal{T}_w$), this is a line bundle on $\mathcal{IW}_w \times_{\mathcal{X}(v)} \mathcal{U}$), and $\omega_w^{\kappa_{\mathcal{U}}^0 \dagger} = \pi_{2*}\mathcal{L}^{\kappa_{\mathcal{U}}}$. Moreover

$$\begin{aligned} R\Gamma(\mathcal{X}^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}\dagger}(-D)) &= R\Gamma(I^0(n), R\Gamma(\mathcal{X}_1(p^n)^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}^0\dagger}(-D))(-\kappa)) \\ &= R\Gamma(I^0(n), R\Gamma(\mathcal{IW}_w \times \mathcal{U}, \mathcal{L}^{\kappa_{\mathcal{U}}}(-D))(-\kappa)). \end{aligned}$$

The last equality is because $\mathcal{IW}_w \longrightarrow \mathcal{X}_1(p^n)^{tor}(v)$ is locally affinoid. Now if you choose $\mathfrak{U}$ a covering of $\mathcal{IW}_w \times \mathcal{U}$ by affinoids which is $I^0(n)$-stable (by adding all translates by $I^0(n)$ if necessary), then the Cech complex of this covering is perfect and calculates $R\Gamma(\mathcal{IW}_w \times \mathcal{U}, \mathcal{L}^{\kappa_{\mathcal{U}}}(-D))$, and twisting the action of $I^0(n)$, and looking at the direct factor of invariants by $I^0(n)$ (we are in characteristic 0), this is still perfect and calculates $R\Gamma(\mathcal{X}^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}\dagger}(-D))$. The same remains true with $\mathcal{X}^{tor}(\underline{\varepsilon})$ (for ε small enough) instead of $\mathcal{X}^{tor}(v)$. □

7. Hecke Operators

In this section we will construct Hecke operators, both at p and outside p. As noted in [**AIP15**, **Bra16**], it is not true that the Hecke correspondences will extend to a fixed choice of a toroidal compactification, nevertheless we can adapt the choice of toroïdal compactifications and use results of Harris ([**Har90a**] Proposition 2.2).

Lemma 7.1 **(Harris).** — *Let Σ, Σ' be two smooth projective polyhedral cone decompositions, and $\mathcal{X}_1(p^n)^{tor}_\Sigma, \mathcal{X}_1(p^n)^{tor}_{\Sigma'}$ the associated toroïdal compactifications. Then there is a canonical isomorphism $H^*(\mathcal{X}_1(p^n)^{tor}_\Sigma(v), \mathcal{O}_{\mathcal{IW}^+}) \simeq H^*(\mathcal{X}_1(p^n)^{tor}_{\Sigma'}(v), \mathcal{O}_{\mathcal{IW}^+})$.*

Proof. — To simplify notation, denote $X_\Sigma = \mathcal{X}_1(p^n)^{tor}_\Sigma$. Up to choosing a common refinement of Σ and Σ', we can suppose that Σ' refines Σ and look at the map

$$\pi : X_{\Sigma'} \longrightarrow X_\Sigma.$$

By results of Harris we have $\pi^*\omega_G = \omega_G'$. Moreover, we can take Σ' small enough (which we do) such that it corresponds to a refinement of the auxiliary datum we chose in section 5.2. In particular, on the integral model $\mathfrak{X}_1(p^n)^{tor}_{\Sigma'}$, the groups H_k are given by pullback of those on $\mathfrak{X}_1(p^n)^{tor}_\Sigma$ and thus we have $\pi : \mathcal{X}_1(p^n)^{tor}_{\Sigma'}(v) \longrightarrow \mathcal{X}_1(p^n)^{tor}_\Sigma(v)$. Thus we have a cartesian square,

$$\begin{tikzcd} \mathcal{IW}^+_{\Sigma'} \arrow[r, "i'"] \arrow[d, "\pi'"] & X_{\Sigma'}(v) \arrow[d, "\pi"] \\ \mathcal{IW}^+_\Sigma \arrow[r, "i"] & X_\Sigma(v) \end{tikzcd}$$

Also, by results of Harris ([**Har90b**] (2.4.3)-(2.4.6)), we have quasi-isomorphisms $\pi_*\mathcal{O}_{X_{\Sigma'}} \xrightarrow{\sim} R\pi_*\mathcal{O}_{X_{\Sigma'}} = \mathcal{O}_{X_\Sigma}$. As $\mathcal{IW}^+ \longrightarrow X(v)$ is flat and π is proper, we have thus by base change (see e.g. [**Sta18**, 30.5.2, Tag 02KH])

$$R\pi'_*\mathcal{O}_{\mathcal{IW}^+_{\Sigma'}} \simeq \mathcal{O}_{\mathcal{IW}^+_{\Sigma}}. \qquad \square$$

7.1. Hecke Operator outside p. — Let λ be a place where our fixed level K^p is hyperspecial, and fix $\gamma \in G(F_\lambda^+)$. Denote C_γ the (analytic space associated to the) moduli space classifiying tuples $(A_k, \iota_k, \lambda_k, \eta_k)$, $k = 1, 2$, of the type G, together with an isogeny $f : A_1 \longrightarrow A_2$ of type γ which respects the additional structure. It is endowed with two maps,

$$C_\gamma \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{X}(v),$$

where $p_k(f : A_1 \longrightarrow A_2) = A_k$. Denote $C_\gamma(p^n) = C_\gamma \times_{\mathcal{X}(v)} \mathcal{X}_1(p^n)(v)$. But we can find choices of smooth projective polyhedral cone decompositions (see [**Lan13**] proposition 6.4.3.4) Σ and Σ' and associated toroïdal compactifications $X^{tor}_\Sigma, C_{\gamma,\Sigma}, X^{tor}_{\Sigma'}$ and maps $p_1 : C_{\gamma,\Sigma} \longrightarrow X^{tor}_\Sigma, p_2 : C_{\gamma,\Sigma} \longrightarrow X^{tor}_{\Sigma'}$ which extends the previous ones. As v is away from p, this correspondence preserves $\mathcal{X}^{tor}(v)$, and the universal isogeny induces an isomorphism,

$$f^* : p_2^*\mathcal{F}_{\mathcal{X}_1(p^n)^{tor}_{\Sigma'}(v)} \xrightarrow{\sim} p_1^*\mathcal{F}_{\mathcal{X}_1(p^n)^{tor}_{\Sigma}(v)},$$

and we can thus construct,

$$H^0(\mathcal{X}_1(p^n)^{tor}_{\Sigma'}(v), \mathcal{O}_{\mathcal{IW}^+}) \xrightarrow{p_2^*} H^0(C_{\gamma,\Sigma}(p^n), p_2^*\mathcal{O}_{\mathcal{IW}^+}) \xrightarrow{f^*}$$
$$H^0(C_{\gamma,\Sigma}(p^n), p_1^*\mathcal{O}_{\mathcal{IW}^+}) \xrightarrow{\mathrm{Tr}\, p_1^*} H^0(\mathcal{X}_1(p^n)^{tor}_{\Sigma}(v), \mathcal{O}_{\mathcal{IW}^+})$$

As by the previous lemma, $H^0(\mathcal{X}_1(p^n)^{tor}_{\Sigma'}(v), \mathcal{O}_{\mathcal{IW}^+}) = H^0(\mathcal{X}_1(p^n)^{tor}_{\Sigma}(v), \mathcal{O}_{\mathcal{IW}^+})$, we get an operator T_γ on $H^0(\mathcal{X}_1(p^n)^{tor}_{\Sigma}(v), \mathcal{O}_{\mathcal{IW}^+})$. Similarly T_γ acts on $C_{cusp}(v, w, \kappa_{\mathcal{U}})$ and $C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{U}})$ (as the isogeny is outside p) and their non-cuspidal analogues. We can thus forget about the choice of Σ in the notations.

Remark 7.2. — Here we made the slight but usual abuse, as we used the notations with a fixed (i, j). Of course, taking tensor products over the (i, j) of the correspondings $\mathcal{IW}^+$ (which depends of the choice of (i, j)) solves this abuse of notation.

Definition 7.3. — Let $\kappa \in \mathcal{W}(w)(L)$ with $w \in]n-1, n-\varepsilon_\tau[$. Restricting the previous operator to homogenous functions on $\mathcal{X}(v)$ for κ, we get the Hecke operator,

$$T_\gamma : M_w^{\kappa\dagger} \longrightarrow M_w^{\kappa\dagger}.$$

Working over $\mathcal{X}(v) \times \mathcal{W}(w)$, considering $\kappa = \kappa^{univ}$, we get an operator

$$T_\gamma^{univ} : M_w^{\kappa^{univ}\dagger} \longrightarrow M_w^{\kappa^{univ}\dagger},$$

which is $\mathcal{O}_{\mathcal{W}(w)}$-linear, and an operator on $C_{cusp}(v, w, \kappa_{\mathcal{W}(w)})$ and $C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{W}(w)})$.

Denote by $\mathcal{H}^{p,S}$ the spherical Hecke algebra of level $K^{p,S}$, the previous construction endow for each w the modules $M_w^{\kappa^{univ}\dagger}$ (respectively the module $M_w^{\kappa\dagger}$ with $\kappa \in \mathcal{W}(w)$) with an action of $\mathcal{H}^{p,S}$.

7.2. Hecke operators at p. — At p, the construction of Hecke Operators is much more subtle than outside p, and even more subtle than in the ordinary case, as already remarked in [**Her19**]. Indeed, when the ordinary locus is non empty, only one operator, $U_{p,g}$ in [**AIP15**], is compact on classical forms (it improves the "Hasse"-radius, i.e. the Hasse invariant), but does not improves the analycity radius for overconvergent forms, whereas the other operators, $U_{p,i}, i < g$ in [**AIP15**], improves (a priori) only the analycity radius. Already for $U(2,1)$ with p inert in the quadratic imaginary field the situation is different. Indeed, there is only one interesting operator, U_p, that improves at the same time both the Hasse-radius and the analycity radius.

Following [**Bij16**], we define operators at p.

7.2.1. *Linear case.* — This is actually easier than the unitary case, and can be adapted from [**Bij16**] on ω^κ to $\omega_w^{\kappa\dagger}$ (in particular there is no p^{-2} appearing in the normalisation corresponding to equation 4, see Remark 8.3). But as this case can be recovered from the general Unitary case (considering $G \times G^D$ with canonical polarisation instead of G), we just write the details in the unitary case.

7.2.2. *Unitary case.* — Fix as before (i,j) a prime, that we supress from now on from the notations, and we can thus use i as a variable. Let G be the associated p-divisible group. Let $0 \leqslant i \leqslant \frac{h}{2}$ an integer, and define C_i the moduli space $(A, \iota, \lambda, \eta, H_\bullet, L)$ where $(A, \iota, \lambda, \eta, H_\bullet) \in \mathcal{X}(v)$ and $L \subset G[p^2]$ be a totally isotropic subgroup such that $H_i \oplus L[p] = G[p]$ and $H_i^\perp \oplus pL = G[p]$, and denote the two maps,

$$C_i \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{X}(v),$$

where $p_1(A,L) = A$ and $p_2(A,L) = A/L$. Denote $C_i(p^n) = C_i \times_{\mathcal{X}(v)} \mathcal{X}_1(p^n)(v)$, and denote $f : A \longrightarrow A/L$ the universal isogeny. As we are in characteristic zero, we can find smooth projective polyhedral cone decompositions $\Sigma, \Sigma', \Sigma''$ such that the previous correspondence extends to $p_1 : C_{i,\Sigma'} \longrightarrow \mathcal{X}^{tor}_\Sigma, p_2 : C_{i,\Sigma'} \longrightarrow \mathcal{X}^{tor}_{\Sigma''}$. In [**Bij16**] Proposition 2.11, Bijakowski verifies that the previous correspondence stabilizes the open $\mathcal{X}(\varepsilon)$. More precisely, he verifies that the Hecke correspondence $U_i = p_2 \circ p_1^{-1}$ satisfies $\deg H_j' \geqslant \deg H_j$ with equality for $i = j$ if and only if $\deg H_i$ is an integer. Its proof extend to the case of a 1-motive in case of bad reduction, and thus extend to the boundary.

In particular, if $\varepsilon_\tau < 1$, by quasi-compacity of

$$\mathcal{X}(\forall \tau, \deg(H_\tau) \in [\lambda_\tau, \nu_\tau]),$$

with

$$\sum_{\tau'} \min(p_{\tau'}, p_\tau) - 1 < \lambda_\tau < \nu_\tau < \sum_{\tau'} \min(p_{\tau'}, p_\tau), \lambda_\tau, \nu_\tau \in \mathbb{Q},$$

we can thus prove the following,

Proposition 7.4. — *For all $w > 0$, for all $\underline{\varepsilon} > 0$ sufficiently small, there exists $\underline{\varepsilon}' < \underline{\varepsilon}$ such that the Hecke correspondence $\prod(U_i)$ sends $\mathcal{X}^{tor}_\Sigma(\underline{\varepsilon})$ into $\mathcal{X}^{tor}_{\Sigma''}(\underline{\varepsilon}')$. Also, for all $\underline{\varepsilon} > 0$, and all $0 < \underline{\varepsilon}' < \underline{\varepsilon}$, there exists $N > 0$ such that $\prod_i U_i^N$ sends $\mathcal{X}^{tor}_\Sigma(\underline{\varepsilon})$ in $\mathcal{X}^{tor}_{\Sigma''}(\underline{\varepsilon}')$.*

The universal isogeny f induces a map,

$$f^* : p_2^* \mathcal{T}_{an} \longrightarrow p_1^* \mathcal{T}_{an},$$

which is an isomorphism, and denote $\widetilde{f}^* = \bigoplus_\sigma \widetilde{f}^*_\sigma$ (using the decomposition $\omega = \bigoplus_\sigma \omega_{G,\sigma}$) such that $\widetilde{f}^*_\sigma$ sends a basis $w'_1, \dots, w'_{p_\sigma}$ of $\omega_{A/L,\sigma}$ to

$$p^{-2}f^*w'_1, \dots, p^{-2}f^*w'_{p_\sigma-h+i}, p^{-1}f^*w'_{p_\sigma-h+i+1}, \dots, p^{-1}f^*w'_{p_\sigma-i}, f^*w'_{p_\sigma-i+1}, \dots, f^*w'_{p_\sigma},$$

(being understood that the terms on the left with p^{-2} only appears if $p_\sigma > h-i$ and terms with p^{-1} only if $p_\sigma > i$). Another way to write it is to set, $a_\sigma = \max(p_\sigma - (h-i), 0), b_\sigma = \max(\min(h-2i, p_\sigma - i), 0)$ and $c_\sigma = \min(i, p_\sigma)$ (thus $a_\sigma + b_\sigma + c_\sigma = p_\sigma$). Then $\widetilde{f}^*_\sigma$ sends $w'_1, \dots, w'_{p_\sigma}$ to,

(4)
$$p^{-2}f^*w'_1, \dots, p^{-2}f^*w'_{a_\sigma}, p^{-1}f^*w'_{a_\sigma+1}, \dots, p^{-1}f^*w'_{a_\sigma+b_\sigma}, f^*w'_{a_\sigma+b_\sigma+1}, \dots, f^*w'_{p_\sigma},$$

Remark 7.5. — This normalisation is made in order to make the operator U_i vary in a family (it corrects the multiplication by p that appears on ω if we do the quotient by L). It is related with normalisation of [**Bij16**] for classical sheaves, but it is not exactly the same, see 8.3.

Fix $\varepsilon = \varepsilon_0 = (\varepsilon_\tau)_\tau < \frac{1}{2}$ small enough and assume (ε, v, n) is satisfied.

Definition 7.6. — Let $\underline{w} = (w^\tau_{i,j})_\tau$, such that for all (i,j,τ), $w^\tau_{i,j} \in]0; n-\varepsilon_\tau[$ and define $\mathcal{IW}^{0,+}_{\underline{w}}$ to be the open subspace of $\mathcal{T}^\times/U$ over $\pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$ such that its L-points, for all L over K, is the datum of a $\mathcal{O}_L$-point of $\pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))(\mathcal{O}_L)$, thus in particular an abelian scheme $A/\operatorname{Spec}(\mathcal{O}_L)$ for which $G[p^n]$ has (canonical by Theorem 5.6) filtration by subgroups H_τ, together with a flag $\mathrm{Fil}_\bullet \mathcal{F}_\tau$ for all τ with graded pieces $w^\tau_\bullet$, such that there exists a (polarized) trivialization ψ (as in section 6.1), such that

$$\omega_i^\tau \pmod{\mathrm{Fil}_{i-1}\mathcal{F}_\tau + p^{w_0^\tau}\mathcal{F}_\tau} = \sum_{j \geqslant i} a_{j,i}\,\mathrm{HT}_{\tau,w_0}(e_j),$$

where $w_0^\tau = n - \varepsilon_\tau$ and with $a_{j,i} \in \mathcal{O}_L$ such that, $v(a_{j,i}) \geqslant w^\tau_{j,i}$ if $j > i$ and $v(a_{i,i}-1) \geqslant w^\tau_{i,i}$. We then define as before $\omega^{\kappa\dagger}_{\underline{w}}$ on $\pi_n(\mathcal{X}_0^+(p^n)^{tor}(v))$ for $\min_{i,\tau} w^\tau_{i,i}$-analytic κ.

Remark 7.7. — In the previous definition, if we take $n' \geqslant n$, $w^\tau_{i,j} \in]0, n-\varepsilon_\tau[$ and we make the previous construction over $\pi_{n'}(\mathcal{X}_0^+(p^n)(v))$ for $w_0 = n - \varepsilon_\tau$ or $w_0 = n' - \varepsilon_\tau$, we get the same space. Thus, up to reducing the strict neighborhood, we suppress n from the notation. When $\underline{w}$ is parallel and $n-1 < w < n - \varepsilon_\tau$, then $\omega^{\kappa\dagger}_{\underline{w}} = \omega^{\kappa\dagger}_w$.

Suppose $\underline{w}$ satisfies

$$\begin{cases} 0 < w^\tau_{k,l} < w_0 - 2 & \text{if } a_\tau \neq 0 \\ 0 < w^\tau_{k,l} < w_0 - 1 & \text{if } a_\tau = 0 \text{ and } b_\tau \neq 0 \\ 0 < w^\tau_{k,l} < w_0 & \text{otherwise} \end{cases}$$

Proposition 7.8. — *Let f be the universal isogeny over C_i. Then*

$$(\widetilde{f}^*)^{-1}p_1^*\mathcal{IW}^{0,+}_{\underline{w}} \subset p_2^*\mathcal{IW}^{0,+}_{\underline{w}'},$$

with

$$\underline{w}'^{\sigma}_{k,l} = \begin{cases} w^\sigma_{k,l} + 2 & \textit{if } k > a_\sigma + b_\sigma \textit{ and } l \leqslant a_\sigma \\ w^\sigma_{k,l} + 1 & \textit{if } (b_\sigma + a_\sigma \geqslant k > a_\sigma \textit{ and } l \leqslant a_\sigma), \textit{ or } (b_\sigma + a_\sigma \geqslant l > a_\sigma \textit{ and } k > a_\sigma + b_\sigma) \\ w^\sigma_{k,l} & \textit{otherwise} \end{cases}$$

Proof. — This is similar to 4.2 and [**AIP15**] Proposition 6.2.2.2. Indeed, in the basis given by the "trivialisation" of $(H_{p_k}/H_{p_{k-1}})^D$ on $\mathcal{X}_0^+(p^n)^{tor}(\varepsilon)$, the dual of the morphism $H_{p_\tau} \longrightarrow H'_{p_\tau}$ induced by f,

$$f^D : (H'_{p_\tau})^D \longrightarrow H^D_{p_\tau},$$

is given by $\mathrm{Diag}(p^2, \dots, p^2, p, \dots, p, 1, \dots 1)$, where p^2 appears a_τ-times, p appears b_τ-times and 1 c_τ-times. The rest follows exactly as in [**AIP15**] Proposition 6.2.2.2, as $\pi^*\mathcal{F}'_\tau \supset p\mathcal{F}_\tau$ is $a_\tau = 0$ and $b_\tau \neq 0$, $\pi^*\mathcal{F}'_\tau \supset p^2\mathcal{F}_\tau$ if $a_\tau \neq 0$ and $\mathcal{F}_\tau = \pi^*\mathcal{F}'_\tau$ otherwise. $\square$

We can thus define the operator U_i^0,

$$H^0(\mathcal{X}^{tor}_{\Sigma''}(\underline{\varepsilon}), \omega^{\kappa\dagger}_{\underline{w}'}) \xrightarrow{p_2^*} H^0(C_i, p_2^*\omega^{\kappa\dagger}_{\underline{w}'}) \xrightarrow{\tilde{f}^{-1*}} H^0(C_i, p_1^*\omega^{\kappa\dagger}_{\underline{w}}) \xrightarrow{\frac{1}{p^{n_i}}\mathrm{Tr}_{p_1}} H^0(\mathcal{X}^{tor}_{\Sigma}(\underline{\varepsilon}), \omega^{\kappa\dagger}_{w})$$

and also

$$\begin{aligned} C_{cusp}(\underline{\varepsilon}, \underline{w}', \kappa^{univ}_{\mathcal{W}(w)}) &\xrightarrow{p_2^*} R\Gamma(C_i \times \mathcal{W}(w), p_2^*\omega^{\kappa^{univ}}_{\underline{w}'}\dagger) \xrightarrow{\tilde{f}^{-1*}} R\Gamma(C_i \times \mathcal{W}(w), p_1^*\omega^{\kappa^{univ}}_{\underline{w}}\dagger) \\ &\xrightarrow{\frac{1}{p^{n_i}}\mathrm{Tr}_{p_1}} R\Gamma(\mathcal{X}^{tor}_{\Sigma}(\underline{\varepsilon}) \times \mathcal{W}(w), \omega^{\kappa^{univ}}_{w}\dagger) = C_{cusp}(\underline{\varepsilon}, w, \kappa^{univ}_{\mathcal{W}(w)}), \end{aligned}$$

where n_i is an integer defined in [**Bij16**] section 2.3 for example(20). It is related to the inseparability degree of the projection p_1.

Remark 7.9. —
1. Unfortunately it is not clear how to define the Hecke operator U_i^0 on the neighborhoods $\mathcal{X}(v)$ as we don't know how the Hasse invariants behaves with quotients... But we will solve this in the end of the paper.
2. Thus we can use the different operators U_i^0 to improve the radius of convergence in all directions $w^\tau_{k,l}$ with $k > l$.

7.3. A compact operator. — Using the previous construction, we can define a compact operator. Fix $w > 0$ and n sufficiently big such that $n - 2 - \varepsilon > w$. Fix also v sufficiently small such that (ε, n, v) is satisfied.

Define $w' = (w'^{\sigma}_{k,l})_{\sigma, k > l}$ by,

$$w'^{\sigma}_{k,l} = \begin{cases} w & \text{if } k = l \\ w + 1 & \text{otherwise} \end{cases}$$

Remark 7.10. — We could be more precise about the precise values of w' we can choose for what follows (summing over all i's the previous proposition), but the previous will be sufficient.

(20)For us, this integer will not be important as it is used to normalize the Hecke operators and is a constant of the weight. As our Hecke eigensystems are constructed on spaces where p is inverted, this normalisation could be changed (we should change Theorem 8.4 accordingly)

Denote by $\underline{\varepsilon}' < \underline{\varepsilon}$ the tuple given by Proposition 7.4. Then we have for each $\kappa \in \mathcal{W}(w)$, the following operator,

$$\prod_{i\geqslant 1} U_i^0 : H^0(\mathcal{X}^{tor}(\underline{\varepsilon}'), \omega_{\underline{w}'}^{\kappa\dagger}) \longrightarrow H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_{\underline{w}}^{\kappa\dagger}),$$

and thus the operator,

$$\prod_{i\geqslant 1} U_i : H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_{\underline{w}}^{\kappa\dagger}) \longrightarrow H^0(\mathcal{X}^{tor}(\underline{\varepsilon}'), \omega_{\underline{w}'}^{\kappa\dagger}) \stackrel{\prod_i U_i^0}{\longrightarrow} H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_{\underline{w}}^{\kappa\dagger}),$$

is compact, as the first map is. Indeed, for some ε let $\pi : \mathcal{IW}_{w,\varepsilon}^{0,+} \longrightarrow \mathcal{IW}_{w,\varepsilon}^{0}$ be the rigid open in, respectively, $(\mathcal{T}/U)_{\mathcal{X}^{tor}(\varepsilon)} \times \mathcal{W}$ and $(\mathcal{T}/B)_{\mathcal{X}^{tor}(\varepsilon)} \times \mathcal{W}$ as in Proposition 6.11, and denote $\mathcal{F}_w = \pi_*\mathcal{O}_{\mathcal{IW}_{w,\varepsilon}^{0,+}}[\kappa^{0,univ}]$, which is an invertible sheaf on $\mathcal{IW}_{w,\varepsilon}^{0}$, and $H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_{\underline{w}}^{\kappa\dagger})$ is a direct factor of $H^0(\mathcal{IW}_{w,\varepsilon}^{0}, \mathcal{F}_w)$. Clearly, as $w' > w$, and $\varepsilon' < \varepsilon$ we have a map

$$res : H^0(\mathcal{IW}_{w,\varepsilon}^{0}, \mathcal{F}_w) \longrightarrow H^0(\mathcal{IW}_{w',\varepsilon'}^{0}, \mathcal{F}_{w'}),$$

and it is enough to show it is compact. By [**KL05**] Proposition 2.4.1, it is enough to show that $\mathcal{IW}_{w',\varepsilon'}^{0} \Subset_{\mathcal{W}} \mathcal{IW}_{w,\varepsilon}^{0}$. But now, e.g. [**KL05**] Proposition 2.3.1, this is reduced to show $\mathcal{IW}_{w,\varepsilon} \Subset_{\mathcal{W}} (\mathcal{T}/B)_{\mathcal{X}^{tor}} \times \mathcal{W}$, which is true as $\mathcal{T}/B \times \mathcal{X}^{tor}$ is proper.

Similarly, denote U_i by precomposing U_i^0 by the map $H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_{\underline{w}}^{\kappa\dagger}) \subset H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega_{\underline{w}'}^{\kappa\dagger})$ of the previous subsection. The same construction works also over $\mathcal{X}^{tor}(\underline{\varepsilon}) \times \mathcal{W}(w)$ with κ^{univ}.

Definition 7.11. — We define $\mathcal{A}(p)^+$ as the commutative $\mathbb{Q}_p$-algebra generated by indeterminates U_i'. Then $\mathcal{A}(p)^+$ acts on $H^0(\mathcal{X}^{tor}(\varepsilon), \omega_{\underline{w}}^{\kappa\dagger})$ for all κ and ε, and also on $H^0(\mathcal{X}^{tor}(\varepsilon) \times \mathcal{W}(w), \omega_w^{\kappa_{\mathcal{W}(w)}\dagger})$, $C(\varepsilon, w, \kappa_{\mathcal{W}(w)})$, and their cuspidal variants, by letting U_i' acts as U_i. Denote $\mathcal{A}(p)$ the $\mathbb{Q}_p$-algebra generated by U_i' and their inverses. Similarly define $\mathcal{H}^{Sp}$ the (spherical) Hecke algebra outside S, the set of places of K of bad reduction, and p. Then the previous sections induces an action of $\mathcal{H}^{Sp}$ on all these spaces.

Remark 7.12. — In the linear case, we can see $\mathcal{A}(p)^+$ as the (commutative !) $\mathbb{Q}_p$-algebra generated by

$$\begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix}, \quad a_1 \geqslant \cdots \geqslant a_n,$$

where we use a choice $w|v|\ p$ in F_i to identify $G(\mathbb{Z}_p)$ with $\mathrm{GL}_n((F_i^+)_v)$. See [**BC09**] Section 6.4 (but our $\mathcal{A}(p)^+$ is $\mathcal{A}(p)^-$ there). Here we like to see the Iwahori level using the upper triangular Borel. Then U_i corresponds to

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & p^{-1} & & \\ & & & & \ddots & \\ & & & & & p^{-1} \end{pmatrix},$$

where 1 appears i times and p^{-1} appears $n-i$ times. We can also see this $\mathcal{A}(p)^+$ algebra as acting on π^I, where π is a $G(\mathbb{Q}_p)$-representation and the action of the previous diagonal matrix d is by $[IdI]$ in the classical Hecke algebra for I (which is commutative). There is a similar description in the unitary case. As $\mathcal{A}(p)$ is constructed by adding inverses to U_i', this will act on the finite slope part in the coherent cohomology, but careful that inverses of elements of $\mathcal{A}(p)^+$ does not acts as the corresponding double class in the classical Hecke algebra !

8. Classicity results

Let w and $\underline{\varepsilon}$ such that $0 < w < 1 - \underline{\varepsilon}$. Up to reduce $\underline{\varepsilon}$ this is always possible for some $w < 1$. Then the map $\mathcal{IW}_w^0 \longrightarrow \mathcal{X}^{tor}(\underline{\varepsilon})$ has connected fibers. Denote $\omega_{w^-}^{\kappa\dagger} := \varinjlim_{w'<w} \omega_{w'}^{\kappa\dagger}$.

Proposition 8.1. — *Let κ be a classical weight. We have an exact sequences of sheaves over $\mathcal{X}^{tor}(\underline{\varepsilon})$,*

$$0 \longrightarrow \omega^\kappa \longrightarrow \omega_w^{\kappa\dagger} \xrightarrow{d_1} \bigoplus_{\alpha\in\Delta} \omega_{w^-}^{\alpha\cdot\kappa\dagger}$$

which etale locally is isomorphic to the exact sequence (2).

Proof. — We construct the map d_1 as in [**AIP15**] (we don't need the hypothesis on w here). Then we have a sequence,

$$0 \longrightarrow V_\kappa \hat{\otimes} \mathcal{O}_{\mathcal{X}^{tor}(\underline{\varepsilon})} \longrightarrow V_{\kappa,L}^{0,w-an} \hat{\otimes} \mathcal{O}_{\mathcal{X}^{tor}(\underline{\varepsilon})} \xrightarrow{d_1} \bigoplus_{\alpha\in\Delta} V_{\alpha\cdot\kappa,L}^{0,w^- - an} \hat{\otimes} \mathcal{O}_{\mathcal{X}^{tor}(\underline{\varepsilon})},$$

which is exact by hypothesis on w by Jones result (hypothesis implies that $N_1 \simeq \mathbb{B}$ as in [**Jon11**] section 8). Then as this sequence is etale locally isomorphic to the one of the proposition, we get the result. □

Proposition 8.2. — *Let $\kappa = (k_{\sigma,j})_{\sigma,1\leqslant j\leqslant p_\sigma}$ be a classical weight. The submodule of $M_w^{\kappa\dagger}(\mathcal{X}^{tor}(\underline{\varepsilon}))$ on which each U_i acts with slope strictly less than* $\inf\{k_{\sigma,i} - k_{\sigma,i+1} : i < p_\sigma\}$ *is contained in $H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega^\kappa)$.*

Proof. — By the previous proposition, and Proposition 7.4 and Proposition 7.8, the proof is identical to [**AIP15**] Proposition 7.3.1. Indeed, let $f \in M_w^{\kappa\dagger}(\mathcal{X}^{tor}(\underline{\varepsilon}))$ on which the U_i acts with the said slope. Using Proposition 7.8, and that f is finite slope for U_i, we can assume that $\underline{w} < 1 - \underline{\varepsilon}$. Thus, by proposition 8.1, because of the slope, we calculate as in [**AIP15**] that $d_1 f = 0$. Thus $f \in H^0(\mathcal{X}^{tor}(\underline{\varepsilon}), \omega^\kappa)$. □

Remark 8.3. — Let us make explicit the normalisation of our Hecke operators, in particular the effect of the operation in equation (4). In case (AL), for $i \in \{0, \dots, h\}$, we choose $L_0 \subset G[p]$ such that $L_0 \oplus H_i = G[p]$ and we have an isogeny

$$f : G \times G^D \longrightarrow G/L_0 \times G^D/L_0^\perp.$$

This induces a map

$$f^* : p_2^* \mathcal{T}_{an} \longrightarrow p_1^* \mathcal{T}_{an},$$

sending, for $\tau \in \mathcal{T}$ a basis $(w_1^\tau, \dots, w_{p_\tau}^\tau)$ of $\omega_{G/L_0,\tau}$[21] to its image via

$$\omega_{G/L_0} \oplus \omega_{G^D/L_0^\perp} \longrightarrow \omega_G \oplus \omega_{G^D}.$$

We then denote $\tilde{f}^*$ which sends a basis $(w_1^\tau, \dots, w_{p_\tau}^\tau)$ to

$$(p^{-1} f^* \omega_1^\tau, \dots, p^{-1} f^* \omega_{p_\tau - i}^\tau, \omega_{p_\tau - i+1}^\tau, \dots, \omega_{p_\tau}^\tau).$$

Seeing $\tilde{f}^*$ as a morphism on functions on trivialisation of ω, this means that f^* sends a trivialisation ψ of $p_2^*\omega$ to $\pi^* \circ \psi(\cdot d_i)$, where d_i is in entry τ the matrix of size p_τ

$$\begin{pmatrix} p^{-1} & & & & & \\ & \ddots & & & & \\ & & p^{-1} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

with $\max(p_\tau - i, 0)$ times p^{-1} appearing. If $\kappa = (k_{\tau,i})_{\tau \in \mathcal{T}, 1 \leqslant i \leqslant p_\tau}$ is a classical weight, this induces a normalisation by a factor

$$p^{-\sum_{\tau \in \mathcal{T}} k_{\tau,p_\tau} + \dots + k_{\tau,i+1}},$$

with the obvious abuse of notation. In case (AU) this is slightly more complicated but $\tilde{f}^*$ sends a trivialisation ψ to $\pi^* \circ \psi(\cdot d_i)$, where d_i is the matrix with τ-entry (of size p_τ) given by

$$\begin{pmatrix} p^{-2} & & & & & & & & \\ & \ddots & & & & & & & \\ & & p^{-2} & & & & & & \\ & & & p^{-1} & & & & & \\ & & & & \ddots & & & & \\ & & & & & p^{-1} & & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{pmatrix}$$

[21] Recall that in case AL we have chosen a section $\mathcal{T}^+ \subset \mathcal{T}$, i.e. a CM type induced by the choice of a place, and if $\tau \in \mathcal{T}^+$ we denote $\omega_{G,\overline{\tau}} := \omega_{G^D,\tau}$ thus here $\omega_{G/L_0,\overline{\tau}} = \omega_{G^D/L_0^\perp,\tau}$.

with p^{-2} appearing a_τ-times, p^{-1} appearing b_τ-times. Remark that if p^{-2} appears for τ, no p^{-2} nor p^{-1} appears for $\overline{\tau}$. In particular for a classical weight κ we get a normalisation by the power of p

$$-\sum_{\tau\in\mathcal{T}} 2(k_{\tau,p_\tau}+\dots+k_{\tau,h-i+1})+(k_{\tau,h-i}+\dots+k_{\tau,i+1}),$$

with the same obvious abuse of notation. In both case we also normalise by p^{-n_i} with n_i independent of the weight (along the trace map).

The second result we need for classicity is Bijakowski's result, [**Bij16**]. For each $\tau\in\mathcal{T}^+$, denote $A_\tau=\min(p_\tau,q_\tau)$ in case (AU) and $A_\tau=p_\tau, \tau\in\mathcal{T}^+$ in case (AL).

Theorem 8.4 **(Bijakowski).** — *Let $\kappa=(k_{\tau,j},\lambda_{\tau,l})_{\tau\in\mathcal{T}^+,1\leqslant j\leqslant p_\tau,1\leqslant l\leqslant q_\tau}$ be a classical weight and let $f\in H^0(\mathcal{X}^{tor}(\underline{\varepsilon}),\omega^\kappa)$. Suppose that f is an eigenvector for the Hecke operators U_{A_τ} of eigenvalue α_τ such that,*

$$n_{A_\tau}+v(\alpha_\tau)<\inf_\tau(k_{\tau,p_\tau}+\lambda_{\tau,q_\tau}),$$

for each $\tau\in\mathcal{T}^+$ verifying $A_\tau\neq 0$. Then f is classical.

Proof. — This is almost exactly Bijakowski's Theorem [**Bij16**, Theorem 4.2], except that his normalisation for the Hecke operators is slightly different (see *loc. cit.* section 2.3). I claim that still with our stronger hypothesis the classicity remains true. The reason is that in Proposition 3.9 and 3.15 of [**Bij16**] we can strengthen bound on the norm of the morphism

$$\omega^\kappa_{A/L}\longrightarrow\omega^\kappa_A.$$

This can be done for each $\tau'\in\mathcal{T}$, and remark that $\omega^{\kappa_{\tau'}}_{A,\tau'}=\omega^{\kappa^0_{\tau'}}_{A,\tau'}\otimes\omega^{\underline{k}_{\tau',p_{\tau'}}}_{A,\tau'}$ with

$$\kappa^0_{\tau'}=(k_{\tau',1}-k_{\tau',p_{\tau'}},\dots,k_{\tau',p_{\tau'}-1}-k_{\tau',p_{\tau'}},0)\quad\text{and}\quad\underline{k}_{\tau',p_{\tau'}}=(\kappa_{\tau',p_{\tau'}},\dots,\kappa_{\tau',p_{\tau'}}).$$

By *loc. cit.* we have a bound on the morphism on $\omega^{\underline{k}_{\tau',p_{\tau'}}}_{A,\tau'}$ related to the degrees of L, so let's give a bound (which will be independent of the degree) for the other part and we even assume that we have any $\kappa=(k_1,\dots,k_p)$ (we don't need the last entry of κ to be zero). Denote $i=p_\tau$ and τ' such that $p=_{\tau'}>p_\tau$, and denote $(A,\lambda,\iota,\eta,H.)$ as in *loc. cit.* an $\mathcal{O}_K$-point of $\mathcal{X}$, and $L_0\subset G[\pi]$ (in case AL) such that $H_i\oplus L_0=G[\pi]$. Then L_0 is of τ'-degree bigger than $p'_\tau-p_\tau$. But looking at

$$\omega_{G/L_0,\tau'}\xrightarrow{M}\omega_{G,\tau'},$$

which is a resolution of $\omega_{L_0,\tau'}$ we have that choosing carefully a basis of both term we can assume that M is given by a $p=p'_\tau$-square matrix

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_p\end{pmatrix}$$

and $v(a_1) + \cdots + v(a_p) = \deg_{\tau'}(L_0) \geqslant p - i$, but $v(a_i) \leqslant 1$ (as $L_0 \subset G[\pi]$). But ω_τ^κ is a submodule of

$$\mathrm{Sym}^{k_1-k_2}\omega_\tau \otimes \mathrm{Sym}^{k_2-k_3}(\bigwedge^2 \omega_\tau) \otimes \cdots \otimes \mathrm{Sym}^{k_{p-1}-k_p}(\bigwedge^{p-1}\omega_\tau) \otimes \det{}^{k_p}\omega_\tau,$$

thus it is enough to prove the result for this vector bundle. But on each term, we have for $r > i = p_\tau$, for each $i_1 < i_2 < \cdots < i_r$,

$$v(a_{i_1} \dots a_{i_r}) \geqslant r - i.$$

Indeed otherwise reduce to $i_j = j$, then as $v(a_k) \leqslant 1$, $v(a_1 \dots a_p) = v(a_1 \dots a_r) + v(a_{r+1}) + \cdots + v(a_p) < r - i + p - r = p - i$, which is absurd. Thus the norm induced by $\omega_{G/L_0} \longrightarrow \omega_G$ is less than

$$p^{-\sum_{r>i}^{p-1}(k_r-k_{r+1})(r-i)-k_p(p-i)} = p^{-k_{i+1}-k_{i+2}-\cdots-k_p}.$$

This is completely analogous for G^D with $L_0^\perp$. In particular using this on $\omega_{A,\tau'}^{\kappa^0_{\tau'}}$ we get that the valuation of the non normalized $(U_i^0)^{bad}$ on weight κ is bigger than

$$\sum_{\tau'|p_{\tau'}>i}(k_{\tau',i+1} - k_{\tau',p_{\tau'}}) + \cdots + (k_{\tau',p_{\tau'}-1} - k_{\tau',p_{\tau'}}) + N_i - n_i - B,$$

using the notations of [**Bij16**, Proposition3.9], but remark that

$$\sum_{\tau'|p_{\tau'}>i}(k_{\tau',i+1} - k_{\tau',p_{\tau'}}) + \cdots + (k_{\tau',p_{\tau'}-1} - k_{\tau',p_{\tau'}}) + N_i = \sum_{\tau'|p_{\tau'}>i} k_{\tau',i+1} + \cdots + k_{\tau',p_{\tau'}},$$

which is exactly our normalization of U_i. Thus we have

$$||\alpha_i^{-1}U_i^{bad}|| \leqslant p^{v(\alpha_i)+n_i-(1-\alpha-2f\varepsilon\inf_{\sigma\in S_2}(k_\sigma+\lambda_\sigma)},$$

with our normalisation for U_i. This is identical in case (AU), and the rest of [**Bij16**] is identical with this modification. $\square$

9. Projectiveness of the modules of overconvergent forms

Recall that we work over $\mathcal{X}^{tor}$, our Shimura variety with Iwahori level at p, and as explained in section 6.3 we have defined for all n and $w \in]n-1, n-\varepsilon_{0,\tau}[$ (where $\underline{\varepsilon}_0$ was small enough), a sheaf $\omega_w^{\kappa\dagger}$ for all w-analytic $\kappa \in \mathcal{W}$, and a universal one $\omega_w^{\kappa^{univ}\dagger}$, both defined on sufficiently small strict neighborhoods of $\mathcal{X}^{tor}(0) = \mathcal{X}^{tor}(\varepsilon = 0)$, the μ-ordinary canonical locus. We have two families of strict neighborhoods of this locus, each having their advantages. In this section, we prove that essentially we have all the advantages (action and compactness of $U = U_p = \prod_{i\geqslant 1} U_i$, the operator of section 7, and vanishing of higher cohomology) on the finite slope part on both kind of strict neighborhoods. In this section, we assume that on the strict neighborhoods we consider we have the sheaves $\omega_w^{\kappa^{univ}\dagger}$, which means concretely that $\underline{\varepsilon}$ and v are small enough (smaller than a constant which depends on w). Let $\mathcal{U} \subset \mathcal{W}(w) \subset \mathcal{W}$ an open affinoïd such that the universal character $\kappa_{\mathcal{U}}$ is w-analytic.

Proposition 9.1. — *Let $w \leqslant w'$ and $\underline{\varepsilon} \geqslant \underline{\varepsilon}'$. Then the restriction maps,*

$$C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{U}}) \longrightarrow C_{cusp}(\underline{\varepsilon}', w', \kappa_{\mathcal{U}}),$$

are isomorphisms on the finite slope part for $U = \prod_i U_i$. In particular, the finite slope part for U of $C_{cusp}(v, w, \kappa_{\mathcal{U}})$ and $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})$ are the same, and thus are their cohomology groups.

As explained in section 7, it is not clear that $C_{cusp}(v, w, \kappa_{\mathcal{U}})$ or any of its cohomology group is preserved by U. But by proposition 7.4, there exists $N > 0$ an integer (which depends on v à priori) such that U^N preserves $C_{cusp}(v, w, \kappa_{\mathcal{U}})$. We can see that when U acts on a module M, the finite slope part for U of U^N are the same (see for example proof of proposition 9.3). We thus define the finite slope part of $C_{cusp}(v, w, \kappa_{\mathcal{U}})$ for U as the one for U^N. It is then a consequence of the previous equality that the finite slope part of $C_{cusp}(v, w, \kappa_{\mathcal{U}})$ is actually stable by U.

Proof. — Indeed, it is enough to do it for ε' given by proposition 7.4, $w' = w - 1$. We have the factorisation,

$$H^i(C_{cusp}(\underline{\varepsilon}', w', \kappa_{\mathcal{U}})) \xrightarrow{\widetilde{U}} H^i(C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{U}})) \xrightarrow{res} H^i(C_{cusp}(\underline{\varepsilon}', w', \kappa_{\mathcal{U}}))$$

Now for a finite slope section $f \in H^i(C_{cusp}(\underline{\varepsilon}', w', \kappa_{\mathcal{U}}))$, by definition there exists a non zero polynomial P with $P(0) = 0$ and $P(U)f = f$. We can extend f to $H^i(C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{U}}))$ by $P(\widetilde{U})f$. In particular, we can find for all v, an ε and $\varepsilon' \leqslant \varepsilon$ such that,

$$\mathcal{X}^{tor}(\varepsilon') \subset \mathcal{X}^{tor}(v) \subset \mathcal{X}^{tor}(\varepsilon),$$

and the composed restriction map,

$$C(\varepsilon, w, \kappa_{\mathcal{U}}) \longrightarrow C(v, w, \kappa_{\mathcal{U}}) \longrightarrow C(\varepsilon', w, \kappa_{\mathcal{U}}),$$

is an isomorphism on the finite slope part, in particular, $C(v, w, \kappa_{\mathcal{U}})^{fs} = C(\underline{\varepsilon}, w, \kappa_{\mathcal{U}})^{fs}$ and thus these spaces are stable by U. □

In particular, we get

Proposition 9.2. — *$C_{cusp}(v, w, \kappa_{\mathcal{U}})$ has cohomology concentrated in degree zero, and the finite slope part of the cohomology of $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})$ is concentrated in degree zero.*

Proof. — The first part is appendix Theorem A.6. Fix i, then we have restriction maps

$$H^i(C_{cusp}(\underline{\varepsilon}, w, \kappa_{\mathcal{U}})) \longrightarrow H^i(C_{cusp}(v, w, \kappa_{\mathcal{U}})) \longrightarrow H^i(C_{cusp}(\underline{\varepsilon}', w, \kappa_{\mathcal{U}})),$$

(for well chosen $\underline{\varepsilon}, \underline{\varepsilon}', v$) whose composite is an isomorphism on finite slope parts, and the middle module vanishes for $i > 0$. □

According to [**Urb11**] section 2.3.10, we can form the alternated Fredholm determinant,

$$\det(1 - XU | C(\varepsilon, w, \kappa_{\mathcal{U}})).$$

But, because of the results of the previous section, this alternated determinant should actually only be the one in degree 0. Moreover, we will be able to restrict (locally) to the classical construction on an eigenvariety as in [**Col97b**, **Buz07**, **AIP15**].

For this, fix ε, v, w and $\mathcal{U}$ accordingly. By Proposition 6.19, recall that $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})$ and $C_{cusp}(v, w, \kappa_{\mathcal{U}})$ are perfect complexes (in the sense of Urban [**Urb11**]), and the latter one can be represented by the projective (in the sense of Buzzard [**Buz07**] or [**Urb11**]) module in degree 0 $H^0(\mathcal{X}^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}^{univ\dagger}}(-D))$. The compact operator U acts on $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})$, but not a priori on $C_{cusp}(v, w, \kappa_{\mathcal{U}})$, but by Proposition 5.17 and Proposition 7.4, there exists $\varepsilon' < \varepsilon$ and an integer N, which we fix, such that we have inclusions,

$$\mathcal{X}^{tor}(\varepsilon) \subset \mathcal{X}^{tor}(v) \subset \mathcal{X}^{tor}(\varepsilon'),$$

and $U^N(\mathcal{X}^{tor}(\varepsilon)) \subset \mathcal{X}^{tor}(\varepsilon')$. In particular $U^N(\mathcal{X}^{tor}(v)) \subset \mathcal{X}^{tor}(\varepsilon') \subset \mathcal{X}^{tor}(v)$. Thus, U^N is an operator on both $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})$ and $C_{cusp}(v, w, \kappa_{\mathcal{U}})$. We now need to explain how to construct the Eigenvariety. First, we have three Fredholm series over $\mathcal{O}_{\mathcal{U}}$ $F_{U,\varepsilon}$ and $F_{U^N,\varepsilon}$ of U and U^N acting on $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})$, and $F_{U^N,v}$ of U^N acting on $C_{cusp}(v, w, \kappa_{\mathcal{U}}) = H^0(\mathcal{X}^{tor}(v) \times \mathcal{U}, \omega_w^{\kappa_{\mathcal{U}}^{univ\dagger}}(-D))$.

First, we need, as in the classical construction, to do things on a specific cover, so choose a slope covering covering for $\mathcal{U}$ and $F_{U,\varepsilon}$, (V, h) in the sense of definition 2.3.1. of [**JN19**] (this exist, see [**JN19**] Theorem 2.3.2 for example). Over (V, h), we can thus decompose the Fredholm series,

$$F_{U,\varepsilon} = GS,$$

where $G \in \mathcal{O}_V[T]$ is a slope $\leqslant h$ polynomial and $S \in 1 + T\mathcal{O}_V T$ is an entire series of slopes $> h$. Accordingly, by [**Col97b**] Theorem A4.3/5 (or [**JN19**] Theorem 2.2.2), we have slope decompositions for U of complexes,

$$C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}}) = C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})_{U,\leqslant h} \oplus C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})_{U,>h}.$$

Lemma 9.3. — *This decomposition is a slope Nh decomposition for U^N acting on this module, and it induces a slope Nh factorisation of*

$$F_{U^N,\varepsilon} = G'S'.$$

Proof. — We can work on a single module, say M and denote the associated decompositions associated to the slope decomposition of $F_{U,\varepsilon} = GS$,

$$M = M_{U,\leqslant h} \oplus M_{U,>h}.$$

As if $Q(T)$ is a slope $> Nh$ polynomial, the polynomial $Q(T^N)$ is of slope $> h$, we get that U^N is invertible on $M_{U,>h}$. Now let

$$P = \{m \in M |\ \exists Q \text{ of slopes } \leqslant Nh, \text{ such that } Q^*(U^N)m = 0\}.$$

This is clearly a submodule of M, and has if $Q(T)$ has slopes $\leqslant Nh$, $Q(T^N)$ has slopes $\leqslant h$, we have $P \subset M_{U,\leqslant h}$. We claim that $P = M_{U,\leqslant h}$. Denote the $\mathcal{O}_V$-module

$$R = M_{U,\leqslant h}/P.$$

Fix x a point of V, and let $v \in (M_{U,\leqslant h})_x = (M_x)_{U,\leqslant h}$ (which is easily seen to be true, or see e.g. [**JN19**] Theorem 2.2.13), thus if we denote $N(v)$ the sub-$k(x)$-vector space generated by v and its images by U and its powers, $N(v)$ is finite dimensional (say of dimension r). Denote $\mu_{U,v}$ and $\chi_{U,v}$ the minimal and characteristic polynomials of U on $N(v)$. As there exists Q of slopes $\leqslant h$ such that $Q^*(U)$ kills U, $\mu_{U,v}^*$, and thus $\chi_{U,v}^*$ have

$\leqslant h$ slopes. Up to extending scalars, there is a basis of $N(v)$ such that the matrix of U on $N(v)$ is given by

$$\begin{pmatrix} \lambda_1 & & & \star \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_r \end{pmatrix},$$

and we can thus calculate characteristic polynomial of U^N : it is of slope $\leqslant Nh$. By the theorem of Cayley-Hamilton $v \in P$. Thus $R\widehat{\otimes}k(x)$ is zero, and by Nakayama, $R = 0$. In particular we have that $M_{U,\leqslant h} \oplus M_{U,>h}$ is a slope Nh decomposition for U^N, which is functorial with respect to localisations $\mathrm{Spm}(k(x)) \longrightarrow V$ as it comes from the slope decomposition of $F_{U,h}$, thus by **[JN19]** Theorem 2.2.13 it induces a slope Nh decomposition

$$F_{U^N,\varepsilon} = G'S'. \qquad \square$$

Lemma 9.4. — *The restriction map*

$$res : C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}}) \longrightarrow C_{cusp}(v, w, \kappa_{\mathcal{U}}),$$

induces an equality

$$F_{U^N,\varepsilon} = F_{U^N,v}.$$

In particular, over (V, h) *we have a decomposition*

$$C_{cusp}(v, w, \kappa_{\mathcal{U}}) = C_{cusp}(v, w, \kappa_{\mathcal{U}})_{U^N,\leqslant Nh} \oplus C_{cusp}(v, w, \kappa_{\mathcal{U}}))_{U,>Nh},$$

such that, res induces an isomorphism over V*,*

$$C_{cusp}(\varepsilon, w, \kappa_{\mathcal{U}})_{U,\leqslant h} = C_{cusp}(v, w, \kappa_{\mathcal{U}})_{U^N,\leqslant Nh}.$$

Proof. — The first part is because we have a diagram

$$\begin{tikzcd} C_{cusp}(\varepsilon) \arrow[r, "res"] \arrow[d, "U^N"] & C_{cusp}(v) \arrow[d, "U^N"] \arrow[dl, "U^N"] \\ C_{cusp}(\varepsilon) \arrow[r, "res"] & C_{cusp}(v) \end{tikzcd}$$

Where the shortened notations speak for themselves, and thus $U^N : C_{cusp}(v) \longrightarrow C_{cusp}(\varepsilon)$ is a link in the sense of **[Buz07]**. Thus the two power series are equal. The rest follows by lemma 9.3 $\square$

In particular, for each $(w, \underline{\varepsilon})$, for w big enough and $\underline{\varepsilon}$ small enough such that for all τ, $w \in]n-1, n-\varepsilon_\tau[$ (which determines a unique integer n), (v doesn't play a role and can always be chosen so that $\mathcal{X}^{tor}(\varepsilon) \supset \mathcal{X}^{tor}(v)$, which we do here), we can construct an Eigenvariety for the tuple

$$(\mathcal{O}_{\mathcal{W}(w)}, C_{cusp}(\varepsilon, w, \kappa_{\mathcal{W}(w)}), \mathcal{H}^N \otimes \mathcal{A}(p), \prod_{v \in S_p, i \geqslant 1} U_{v,i}),$$

as if $C_{cusp}(\varepsilon, w, \kappa_{\mathcal{W}(w)})$ were one projective module. Indeed, locally this can be replaced by $\pi(v)_*\omega_w^{\kappa_{\mathcal{W}(w)}\dagger}(-D)$ where

$$\pi(v) : \mathcal{X}^{tor}(v) \ \times \mathcal{W}(w) \longrightarrow \mathcal{W}(w),$$

and this $\mathcal{O}_{\mathcal{W}(w)}$-module is indeed projective, its finite slope part inherits the action of $U = \prod_{\pi,i} U_{\pi,i}$, and these constructions glue together. Moreover, we have natural maps between them when $(w, \underline{\varepsilon}), (w', \underline{\varepsilon}')$ satisfies $w' \geqslant w$ and $\underline{\varepsilon}' \leqslant \underline{\varepsilon}$.

This is the main ingredient in all the constructions of Eigenvarieties. In particular, we get,

Theorem 9.5. — *Let p be a prime. Fix S_p a set of primes over p (see section 2) unramified in $\mathcal{D}$ and (K_J, J) a type[22] outside S_p, $K \subset \operatorname{Ker} J$, and S^p the set of places away from S_p where K is not maximal. There exists an equidimensional rigid analytic space $\mathcal{E}_{S_p}$, together with a locally finite map,*

$$\mathcal{E}_{S_p} \xrightarrow{w} \mathcal{W}_{S_p},$$

and a Zariski dense subset $\mathcal{Z}$, such that for any $\kappa \in \mathcal{W}(L)$, $w^{-1}(\kappa)$ is in bijection with the eigensystems for $\mathcal{H}^{S_p} \otimes \mathcal{A}(p)$ acting on the space of overconvergent, locally analytic, modular forms of weight κ, type (K_J, J), and finite slope for $\mathcal{A}(p)$. Moreover, $w(\mathcal{Z})$ consists of classical weights and $z \in \mathcal{Z}$ is an Hecke eigensystem for a classical modular form of weight $w(z)$.

Proof. — The construction is classical as soon as we have the previous datum, see [**Col97b**] and [**Buz07**]. Just remark that cutting in the datum the piece of type (K_J, J) is possible as we are in characteristic zero (see [**Her19**], Proposition 9.13). The equidimensionality results follows from the fact that we locally reduce to a single projective module $H^0(\mathcal{X}(\underline{\varepsilon}), \omega_w^{\kappa_U\dagger}(-D))_{U^N,\leqslant h}$ and [**Che04**] Lemme 6.2.10. The set $\mathcal{Z}$ is the set of points of $\mathcal{E}$ which map to a point in $\mathcal{W}$ satisfiying the hypotheses of proposition 8.2 and theorem 8.4. This is (Zariski) dense by [**Che04**] Corollaire 6.4.4. and using that every open of $\mathcal{W}$ contains a point satisfying the previous hypothesis. □

Remark 9.6. — We will always consider the space $\mathcal{E}_{S_p}$ with its reduced structure (see [**Che05**] section 3.6). But in turns out that $\mathcal{E}_{S_p}$ is almost always automatically reduced with the structure given by $\mathcal{H} \otimes \mathcal{A}(p)$. For the eigencurve this is [**CM98**], Proposition 7.4.5., in the quaternionic case see [**Che05**] Proposition 4.8, and [**BC09**], section 7.3.6 for a unitary group, compact at infinity. In the next section, we will prove that in the case of $U(2,1)$ this is also true.

10. Some complements for Picard modular forms (especially when $p = 2$)

In a previous article (see [**Her19**]), we constructed the Eigenvariety $\mathcal{E}$ for $U(2,1)_{E/\mathbb{Q}}$ where E is a quadratic imaginary field, under the hypothesis that p was inert (if p splits see [**Bra16**]) so that the ordinary locus is empty, but also that $p \neq 2$, so that we can apply the main theorem of [**Her16**] on the canonical filtration. Theorem 9.5 extends this construction also for $p = 2$, and for E/F a general CM-extension (but we only consider

[22] Here by type we only mean, as in [**Her19**], a compact open subgroup K_J of $G(\mathbb{A}_f^p)$ together with a finite dimensional representation

$F = \mathbb{Q}$ in this section). To fix ideas, we set $\tau, \overline{\tau}$ (or $v, \overline{v}$ if p splits) the places above p, and $p_\tau = p_v = 2, p_{\overline{\tau}} = p_{\overline{v}} = 1$. Classical points on $\mathcal{E}$ correspond to classical forms for $(G)U(2,1)$ with ***classical*** weights given by $\kappa = (k_1 \geqslant k_2, k_3) \in \mathbb{Z}^3$. The corresponding character in $\mathcal{W}$ is given, as explained just before Proposition 6.16, by

$$(x,y) \in \mathcal{O}^\times \times \mathcal{O}^1 \longmapsto \tau(x)^{k_1}\tau(y)^{k_2}\sigma\tau(x)^{k_3}.$$

Recall that $\mathcal{E}$ comes with a map $\mathcal{A}(p) \longrightarrow \mathcal{O}(\mathcal{E})^\times$, and we set[23]

$$F_1 = U_pU_0^{-1}, \text{ if } p \text{ is inert} \quad \begin{cases} F_1 = & U_1U_0^{-1} \\ F_2 = & p^{-1}U_2U_1^{-1} \\ F_3 = & pU_3U_2^{-1} \end{cases} \text{ if } p \text{ splits,}$$

these are the respective (up to a normalisation factor related to the Hodge-Tate weights) images of

$$\begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix}, \quad \begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}_v, \begin{pmatrix} 1 & & \\ & p & \\ & & 1 \end{pmatrix}_v, \begin{pmatrix} 1 & & \\ & 1 & \\ & & p \end{pmatrix}_v,$$

in the Atkin-Lehner algebra $\mathcal{A}(p)$ (which we can see in the Iwahori-Hecke algebra), for some presentation of $G(\mathbb{Z}_p)$. When we write $\begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}$ we really mean

$$\left[I\begin{pmatrix} 1 & & \\ & p^{-1} & \\ & & p^{-1} \end{pmatrix}I\right]\left[I\begin{pmatrix} p^{-1} & & \\ & p^{-1} & \\ & & p^{-1} \end{pmatrix}I\right]^{-1}$$

and not the corresponding double class in the Iwahori Hecke algebra ! Precisely when p splits, if $G = A[v^\infty]$ with $\dim G = 2$, then $U_{v,i}$ the operator defined in section 7.2 with L a subgroup of the form $L_i \oplus L_i^\perp$, and $L_i \subset G[p]$ complementary to H_i the i-th canonical subgroup, such that $\{0\} = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 = G[p]$.

Proposition 10.1. — *The space $\mathcal{E}$ is reduced. This remains true if we had fixed the second weight to $k_2 \in \mathbb{Z}$ on the weight space.*

Proof. — We will use **[Che05]** Proposition 3.9, and we only need to check assumption (SSG) there. Thus, we need to find sufficiently many classical points $k \in \mathcal{W}$ for which the module $M_k^{class} \cap M_k^{\dagger,\leqslant\alpha}$ is semi-simple as an $\mathcal{H}^N \otimes \mathcal{A}(p)$-module. We know already that the space of ***cuspidal*** forms for a group G is semi-simple for the action of $\mathcal{H}^N$ (spherical Hecke operators being auto-adjoint). Thus, we need to treat the action of $\mathcal{A}(p)$. But the action on an automorphic form π of $\mathcal{A}(p)$ determines its refinements. Thus, we only need to prove that we can assure that these refinements are distincts, leading that the action of $\mathcal{A}(p)$ on π_p^I will be semi-simple (I is an Iwahori subgroup). Let $k = (k_1, k_2, k_3) \in \mathcal{W}$ be a classical weight. Fix $\alpha \in \mathbb{R}$, and consider $\mathcal{E}^{\leqslant\alpha}$ the eigenvariety constructed with slopes $\leqslant \alpha$ locally around k. It is locally (on the base) finite over $\mathcal{W}$. As the space $H^0(X, \omega^k(-D))$ is finite dimensional, there is a finite number of classical points f in $\mathcal{E}^{\leqslant\alpha}$ mapping to k (and

[23] U_p is the compact operator, equal to U_1 here, and U_0 was denoted S_p in **[Her19]**

for varying k these are strongly Zariski-dense in $\mathcal{E}^{\leqslant\alpha}$). But the slopes of Hecke operators at p are locally constant, thus for each of these points we can find an open U_f (intersecting every component of $\mathcal{E}^{\leqslant\alpha}$ at f) on which the slope is actually constant. As $\mathcal{E}^{\leqslant\alpha}$ is finite above some affinoid U with $x \in U \subset \mathcal{W}$, taking the intersection of the image of U_f by π in $\mathcal{W}$, we can find an open $V \ni k$ in $\mathcal{W}$ and for which every classical point $k' \in V$ and every classical f' in the fiber of k' in $\mathcal{E}^{\leqslant\alpha}$ has slopes equal the same as the one of some classical f in the fiber of k. But the refinements are given (up to twists) in terms of eigenvalues of Frobenius by (see section 7.2 and see also [**Her19**] section 10.6 but normalisations are slightly different)

$$(p^{-(k_1+k_3)}F_1, 1, p^{k_1+k_3}F_1^{-1}) \quad \text{if } p \text{ is inert},$$

with $F_1 = U_p S_p^{-1} \in \mathcal{O}(\mathcal{E})^\times$ and

$$(p^{-k_1}F_1, p^{1-k_2}F_2, p^{-1-k_3}F_3) \quad \text{if } p \text{ splits},$$

where $F_i \in \mathcal{O}(\mathcal{E})^\times$are defined above. In particular $U_{v,1}$ corresponds up to normalisation by p^{-k_2} to the double class

$$\begin{pmatrix} 1 & & \\ & p^{-1} & \\ & & p^{-1} \end{pmatrix}_v, \begin{pmatrix} p^{-1} & & \\ & 1 & \\ & & 1 \end{pmatrix}_{\overline{v}} \in GU(2,1)(\mathbb{Q}_p) \subset \mathrm{GL}_3(E_v) \times \mathrm{GL}_3(E_{\overline{v}}).$$

The normalisation by the weight in both cases arise because of definition of $\tilde{f}^*$ in equation (4). In particular, as the slopes of F_i are constant on $V \ni k$, for any $k' \in V$ with sufficiently regular weights, the three Frobenius eigenvalues are distincts, thus as are the possible refinements. In particular for those k' (which are Zariski dense in $\mathcal{W}$) the action of $\mathcal{H} \otimes \mathcal{A}(p)$ is semi-simple on classical forms in $M_{k'}^{class}$. The same proof works if k_2 is fixed. □

Remark 10.2. —

1. In particular, by [**Che05**] this proves that for a classical $k_2 \in \mathbb{Z}$, $\mathcal{E}'$ given by the full eigenvariety $\mathcal{E}$, base changed over

$$\mathcal{W}_{\mathcal{O}^\times} \hookrightarrow \mathcal{W}, \quad (k_1, k_3) \mapsto (k_1, k_2, k_3),$$

   and the surface constructed as in the previous section, over $\mathcal{W}_{\mathcal{O}^\times}$ with a fixed value for k_2 coincide and are reduced.
2. Obviously, the same result where we would suppose $k_1 = k_3$ would not be true anymore as it could be that there isn't enough classical semi-simple points.

In [**Her19**] (Theorem 1.3), we proved the following theorem,

Theorem 10.3. — *Let $E/\mathbb{Q}$ be a quadratic extension, and*

$$\chi : \mathbb{A}_E^\times / E^\times \longrightarrow \mathbb{C}^\times,$$

an algebraic Hecke character. We suppose χ polarized (i.e. $\chi^\perp := (\chi \circ c)^{-1} = \chi|.|^{-1}$ where c is the complex conjugation on E). Let p be a prime such that p is unramified in E and $p \nmid \mathrm{Cond}(\chi)$, and $p \neq 2$ if p is inert in E. Let

$$\chi_p : G_E \longrightarrow \overline{\mathbb{Q}_p}^\times,$$

be its p-adic realisation. Then, if $\operatorname{ord}_{s=0} L(\chi, s)$ *is even and non-zero, the Bloch-Kato Selmer group* $H^1_f(E, \chi_p)$ *is non-zero.*

This result (actually a more general version of it) was almost entirely already proved by Rubin ([**Rub91**]) at least for CM elliptic curves when $p \neq 2$ (and $p \neq 3$ for $E = \mathbb{Q}(i\sqrt{3})$). In particular as 2 is inert in $\mathbb{Q}(i\sqrt{3})$ (and 3 is ramified in this case), it does not prove anything new for inert primes (only for the split ones above 2 in other quadratic extensions).

Fortunately, with Theorem 9.5, we will be able to remove the hypothesis $p \neq 2$ if inert. Moreover, we can also remove the hypothesis $p \nmid \operatorname{Cond}(\chi)$ (as long as p stays unramified in E).

To do this we focus from now on to the case of $U(2,1)_{E/\mathbb{Q}}$, and let

$$\chi : \mathbb{A}_E^\times / E^\times \longrightarrow \mathbb{C}^\times,$$

be an algebraic Hecke character, that we assume to be polarised. We often identify it with its p-adic realisation and let $\overline{\chi} := \chi^c$. Denote $\chi_\infty(z) = \tau(z)^a \overline{\tau}(z)^b$, then $b = 1 - a \in \mathbb{Z}$. Moreover (up to change χ by χ^c) we assume $a \geqslant 1$.

10.1. A remark on $p = 2$. — To construct an integral model for the Picard modular surface, it is needed to choose a lattice for the group $(G)U(2,1)$, as it appeared in $\mathcal{D}$ in section 3. We do as we did in [**Her19**] and choose the lattice $L = \mathcal{O}_E^3 \subset E^3$, stable for the form of matrix (used to define $(G)U(2,1)$) in the canonical basis given by,

$$\psi = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

There is another natural choice, which would be the same lattice but the form

$$\psi' = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}.$$

These two forms are isomorphic over $\mathbb{Z}[1/2]$ but not modulo 2. Moreover, see [**Bel06**] Section 3.1, any abelian scheme of type (2,1) A/S will have a polarized Tate module $(T_\ell(A), q)$, together with the Weil pairing induced by the polarisation isomorphic either to $(\mathcal{O}^3, J)$ or $(\mathcal{O}^3, J')$. Any of these form would give an integral model for the Picard modular surface, not isomorphic modulo 2, and we choose ψ, the first one, to define $U(2,1)_{E/\mathbb{Q},\psi}$ over $\mathbb{Z}$ as in section 3. Apart to construct the Eigenvariety, this choice (for which the construction of the Eigenvariety can be checked to be independant afterwards, even if we don't need this result) will not appear in this section as we work in characteristic zero.

10.2. Removing the hypothesis $p \nmid \operatorname{Cond}(\chi)$. — Recall that in [**Her19**] Section 10, following [**BC04**], we introduced a type (K_J, J) for $J = \operatorname{Cond}(\chi)$. Fix an auxiliary level $K^p \subset (\operatorname{Ker} J)^p$, and consider $\mathcal{X}_0(p^n)^{tor} / \operatorname{Spm}(K)$ the (rigid and compactified) Picard variety of Iwahori level p^n, over some p-adic field K and fix $\tau : E \longrightarrow K$. It is the analytic space of $X_0(p^n)^{tor}$ which away from the boundary its S-points parametrizes

$$(A, \iota, \lambda, \eta, H_1 \subset H_2),$$

where

- $A \longrightarrow S$ is an abelian scheme of genus 3
- $\iota : \mathcal{O}_E \hookrightarrow End_S(A)$ is a CM-structure of signature (2,1), i.e.

$$\omega_A = \omega_{A,\tau} \bigoplus \omega_{A,\overline{\tau}}, \quad \omega_{A,\sigma} = \{w \in \omega_A | \iota(x)w = \tau(x)w,$$

with $\omega_{A,\tau}$ and $\omega_{A,\overline{\tau}}$ are respectivelly locally free of rank 2 and 1, and where $\tau : E \longrightarrow S$ is the canonical morphism and $\overline{\tau}$ its conjugate.
- $\lambda : A \longrightarrow {}^tA$ is a polarisation for which the Rosati involution on $\iota(x)$ coincides with $\iota(\overline{x})$
- η is a level-K^p-structure,
- $H_1 \subset H_2 \subset A[p^n]$ is a filtration by cyclic $\mathcal{O}_E \otimes \mathbb{Z}_p/p^n\mathbb{Z}_p$-modules such that $H_2^\perp = H_1$.

This is exactly the rigid space introduced in section 5.

The subgroups H_1, H_2 extends to $\mathcal{X}_0(p^n)^{tor}$, and we can also extend the polarisation of H_2/H_1 to the boundary. We will distinguish the cases p inert (AU) and p split (AL) in E.

In case (AL), i.e. $p = v\overline{v}$ is split, then $A[p^n] \simeq G^+ \times G^-$ (with $G^- = (G^+)^D$ and λ exchange the two factors), and we can suppose that $G = G^+$, say, is of dimension 2 and height p^{3n}. Under this decomposition, $H_i = H_i^+ \times H_i^-$ and H_i^+ is a cyclic rank p^{in}-subgroup of G^+ and $H_1^- = (H_2^+)^\perp = (G^+/H_2^+)^D \subset G^-$. In this case

$$\mathcal{X}_0^+(p^n) = \mathrm{Isom}(H_1^+, \mathbb{Z}/p^n\mathbb{Z}) \times \mathrm{Isom}(H_2^+/H_1^+, \mathbb{Z}/p^n\mathbb{Z}) \times \mathrm{Isom}(G^+/H_2^+, \mathbb{Z}/p^n\mathbb{Z}).$$

It is a $T_n = ((\mathbb{Z}/p^n\mathbb{Z})^\times)^3$-etale torsor. Remark that H_2^+ is the canonical subgroup in this case. In case (AL) we can also introduce a second space. Using the previous notation, denote by $\mathcal{X}_{P_n}^{tor}$ the analytic space associated to a toroïdal compactification of the following moduli space $X_{P_n}^{tor}$ over $\mathrm{Spec}(K)$. A S-point of $X_{P_n}^{tor}$ is, away from the boundary, a tuple $(A, \iota, \lambda, \eta, H_1^p, H_2^p, H)$ such that $(A, \iota, \lambda, \eta, H_1^p, H_2^p)$ is a S-point of $X_0(p)$ (Iwahori level, i.e. $H_1^p \subset H_2^p \subset G^+[p]$) together with a subgroup $H \subset G[p^n]$ locally isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^2$ and $H[p] = H_2^p$. It is the Shimura variety of level $P_n \cap I(p)$ where $I(p)$ is the Iwahori subgroup of $GL_3(\mathbb{Z}_v)$ and P_n is the subgroup of matrices of the form

$$\begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ & & \star \end{pmatrix} \pmod{p^n}.$$

In particular we have a map $\mathcal{X}_0(p^n) \longrightarrow \mathcal{X}_{P_n}$.

In case (AU), i.e. p inert, denote

$$\mathcal{X}_0^+(p^n) = \mathrm{Isom}(H_1, \mathcal{O}/p^n\mathcal{O}) \times \mathrm{Isom}_{pol}(H_2/H_1, \mathcal{O}/p^n\mathcal{O}).$$

This is a $T_n = (\mathcal{O}/p^n\mathcal{O})^\times \times (\mathcal{O}/p^n\mathcal{O})^1$-etale torsor.

In both cases, if $\pi : \mathcal{X}_0^+(p^n)^{tor} \longrightarrow \mathcal{X}_0(p^n)^{tor}$ and $\varphi : T_n \longrightarrow K^\times$ is a character, we can consider $\mathcal{O}_{\mathcal{X}_0(p^n)^{tor}}(\varphi)$ to be the subsheaf of $\pi_*\mathcal{O}_{\mathcal{X}_0^+(p^n)^{tor}}$ of sections which vary like φ. This is an invertible sheaf on $\mathcal{X}_0(p^n)^{tor}$.

Definition 10.4. — For all classical weight κ, we can consider the sheaf,

$$\omega^\kappa(\varphi) := \omega^\kappa \otimes_{\mathcal{O}_{\mathcal{X}_0(p^n)^{tor}}} \mathcal{O}_{\mathcal{X}_0(p^n)^{tor}}(\varphi),$$

which is a locally free sheaf on $\mathcal{X}_0(p^n)^{tor}$, whose global sections are (classical) Picard modular form of weight κ and nebentypus φ. Similarly,

$$H^0(\mathcal{X}_0(p^n)^{tor}, \omega^\kappa(\varphi)(-D)),$$

is the set of cuspidal ones.

Proposition 10.5. — *There is a natural injection*

$$\omega^\kappa(\varphi) \hookrightarrow \omega_w^{\kappa\varphi\dagger},$$

for all $w \in]n-1, n-\varepsilon_\tau[$ and $\kappa\varphi$ the product of the character κ with the character

$$T(\mathbb{Z}_p) \longrightarrow T_n \xrightarrow{\varphi} K^\times,$$

which we still denote φ.

Proof. — Indeed, a section f of $\omega^\kappa(\varphi)$ is a law which associate to (A, x, w) where $A \in \mathcal{X}^{tor}(K)$, x is a level $X_0^+(p^n)$-structure and w an isomorphism

$$St_{\mathcal{O}_E} \otimes \mathcal{O}_K \simeq \omega_A,$$

an element $f(A, x, w) \in \mathbb{A}^1(K)$, which moreover satisfies,

$$f(A, tx, zw) = \varphi(t)\kappa^\vee(z) f(A, x, w).$$

In particular, this defines by restriction a section g of $\mathcal{IW}_w^{0,+}$ which satisfies, for the induced action of $T(\mathbb{Z}_p)$ on $\mathcal{IW}_w^{0,+}$ (using ι !) which sends (A, x, w) to $(A, \bar{t}x, \iota(t)w)$, such that $g(ti) = \varphi(\bar{t})\kappa(t)g(i)$. Thus g is a section of $\omega^{\kappa\varphi\dagger}$. □

Over $\mathcal{X}_{P_n}^{tor}$ we also have a $I_{GL_2}(p)(\mathbb{Z}/p^n\mathbb{Z}) \times (\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor (where $I_{GL_2}(p)$ is the Iwahori subgroup of $GL_2(\mathbb{Z}_p)$), given by

$$\mathrm{Isom}_{modp}(H, (\mathbb{Z}/p^n\mathbb{Z})^2) \times \mathrm{Isom}(G^+/H, \mathbb{Z}/p^n\mathbb{Z}),$$

where $modp$ means that an isomorphism ϕ induces an isomorphism of H_p^1 inside $p^{n-1}\mathbb{Z}/p^n\mathbb{Z}e_1$. Thus, for φ' a character of $I_{GL_2}(p)(\mathbb{Z}/p^n\mathbb{Z}) \times (\mathbb{Z}/p^n\mathbb{Z})^\times$, i.e. of the form $(\varphi_1 \circ \det, \varphi_3)$, we have an invertible sheaf $\mathcal{O}(\varphi')$ on $\mathcal{X}_{P_n}^{tor}$ and thus a sheaf $\omega^\kappa(\varphi')$. The sheaf $\omega^\kappa(\varphi)$ on $\mathcal{X}_0(p^n)^{tor}$ descend to $\mathcal{X}_{P_n}^{tor}$ if and only if $\varphi = (\varphi_1, \varphi_1, \varphi_3)$ and coincides with $\omega^\kappa(\varphi')$ with $\varphi' = (\varphi_1 \circ \det, \varphi_3)$.

Proposition 10.6. — *In the split case, if $v < \frac{1}{2p^{n-1}}$, the canonical subgroup induces an isomorphism,*

$$\mathcal{X}_0(p)^{tor}(v) \longrightarrow \mathcal{X}_{P_n}^{tor}(v).$$

Here we really mean the μ-canonical locus (and not the full μ-ordinary locus).

Remark 10.7. — If $p > 2$, is inert, the results of **[Her16]** give, for $v < \frac{1}{4p^{n-1}}$, an isomorphism $\mathcal{X}(v) \longrightarrow \mathcal{X}_0(p^n)(v)$.

Recall the following result of Rogawski (see **[BC09]** section 6.9.6 and **[Bel10]** section 2.7). Fix first a Hecke character μ as in **[BC09]** Lemma 6.9.2(iii).

***Theorem 10.8* (Rogawski).** — *Suppose* $\mathrm{ord}_{s=0} L(\chi, s)$ *is even and non zero. Then there exists a representation* π^n*, automorphic for* $U(2,1)$ *and cuspidal such that for every prime* x *split in* E*,*

$$L(\pi^n_x) = \mu|.|^{-\frac{1}{2}}(L(\overline{\chi} \oplus 1 \oplus |.|).$$

Remark 10.9. — This representation π^n is slightly different from the one of [**BC04**] or [**Her19**], it is a twist of the latter by $L(\overline{\chi})\mu|.|^{-\frac{1}{2}}$.

Recall that $\chi : \mathbb{A}_E^\times/E^\times \longrightarrow \mathbb{C}^\times$ is a Hecke character, to which is associated its p-adic representation

$$\chi = \chi_p : G_E \longrightarrow \overline{\mathbb{Q}_p}^\times.$$

We hope that the context is sufficiently clear to know which we refer to when writing χ. To avoid confusion, we denote χ_p^{sm} the (smooth) component at p of the adelic χ.

Proposition 10.10. — *Suppose* $\mathrm{ord}_{s=0} L(\chi, s)$ *is even and non zero. Denote* $n_0 = v_p(\mathrm{Cond}(\chi))$*. Denote by* φ *if* p *is split the character* $(1, 1, (\chi_p^{sm})^{-1})$ *and if* p *is inert the character* $((\chi_p^{sm})^{-1}, (\chi_p^{sm})^{-1})$ *of* $(\mathcal{O}/p^n\mathcal{O})^\times \times (\mathcal{O}/p^n\mathcal{O})^1$*. Denote also* κ *the classical weight corresponding to*

$$(1, 2-a, 1) \in \mathbb{Z}^3_{dom}.$$

Then the Hecke eigensystem (away from $p\,\mathrm{Cond}(\chi)$*) of* π^n *appears in* $H^0(\mathcal{X}(\varepsilon), \omega_w^{\kappa\varphi\dagger}(-D))$ *for all* $n \geqslant n_0$ *and* $w \in]n-1, n-\varepsilon[$ *for* ε *small enough.*

Proof. — Indeed we checked that π^n contributes to the coherent first cohomology group in [**Her19**] Proposition D.2. More precisely we checked that its restriction to $SU(2,1)$ appears with K-type corresponding to κ restricted to $SU(2,1)$. As π^n is a twist of the representation denoted $\pi^n(\chi)$ in [**Her19**] by $\overline{\chi}\mu|.|^{-1/2}$, which is algebraic, we can calculate its algebraic weight κ and check that $\kappa = (1, 2-a, 1)$[24]. Moreover Bellaiche-Chenevier ([**BC04**] Proposition 4.2) proved that $\pi^n(\chi) \otimes \chi_0^{-1}$ was of a certain type (K_J, J) at ramified primes for χ. As $\overline{\chi} = \overline{\chi_0}|.|^{\frac{1}{2}} = \chi_0^{-1}|.|^{\frac{1}{2}}$ and both $|.|$ and μ are unramified at p, we deduce that the twist π^n is of the same type as $\pi^n(\chi) \otimes \chi_0^{-1}$ (which is obviously trivial if χ is unramified at p). Thus π^n is of nebentype φ, and we deduce the previous result from proposition 10.5. $\square$

We need to take care of the action at p of the Iwahori algebra $\mathcal{A}(p)$. This is well known in the case of GL_2 (see [**Col97a**]). Denote the higher-Iwahori subgroup

$$I_n^+ = \begin{pmatrix} 1+p^n\mathcal{O} & \star & \star \\ p^n\mathcal{O} & 1+p^n\mathcal{O} & \star \\ p^n\mathcal{O} & p^n\mathcal{O} & 1+p^n\mathcal{O} \end{pmatrix} \cap G(\mathbb{Q}_p),$$

where $G(\mathbb{Q}_p) = \mathrm{GL}_3(\mathbb{Q}_p)$ if p is split, and $U(2,1)(\mathbb{Q}_p) = U(3)(\mathbb{Q}_p)$ otherwise. We could do everything for $GU(2,1)$ or $\mathrm{GL}_3 \times \mathrm{GL}_1$ (if p splits) but it doesn't change anything for

[24] We could also argue directly as in [**Her19**] relating κ to the Hodge-Tate weights of ρ_{π^n} on the Eigenvariety $\mathcal{E}$. Remark that for the τ-Hodge-Tate weight of π^n there is a twist by $1-a$ compared to those of $\pi^n(\chi)$. This is compatible with the twist by $(a-1, a-1, 1-a)$ on the coherent weight κ given in formula before Proposition 10.20

us. I_n^+ has a natural Iwahori decomposition $I_n^+ = \overline{N}_n \times T_n^+ \times N_n$ (and $N_n = N$), and thus if we denote Σ^+ the elements of the form

$$\begin{pmatrix} p^{a_1} & & \\ & p^{a_2} & \\ & & p^{a_3} \end{pmatrix} \quad \text{with } a_1 \geqslant a_2 \geqslant a_3,$$

if p splits, and

$$\begin{pmatrix} p^{a_1} & & \\ & p^{a_2} & \\ & & p^{-a_1} \end{pmatrix} \quad \text{with } a_1 \geqslant a_2,$$

if p is inert. Denote by Σ the group generated by Σ^+ and their inverse.

Proposition 10.11. — *Denote by $\mathcal{A}_n^{+,0}(p)$ the sub-algebra of $\mathcal{H}(G(\mathbb{Q}_p)//I_n^+)$ generated by the double class characteristic functions*

$$1_{I_n^+ a I_n^+}, \quad a \in \Sigma^+.$$

$\mathcal{A}_n^{+,0}(p)$ is commutative. Denote by $\mathcal{A}_n^+(p)$ the algebra generated over $\mathbb{Q}_p$ by $\mathcal{A}_n^{+,0}(p)$ and the inverse of the elements $1_{I_n^+ a I_n^+}$. It is canonically isomorphic to Σ and thus to $\mathcal{A}(p)$.

Proof. — $\mathcal{A}_n^+(p)$ is commutative by [**Cas95**] Lemma 4.1.5. □

Remark 10.12. — The canonical isomorphism $\Sigma \longrightarrow \mathcal{A}_n^+(p)$ sends $a \in \Sigma^+$ to the corresponding double class, but this is not true for all $a \in \Sigma$, just like the case of $\mathcal{A}(p)$. The double class are not invertible in general (if $n > 1$ at least, see [**Ogg69**] Lemma 2 for (new) modular forms, but this is true if $n = 1$, [**Vig16**]).

There is thus an Hecke operator acting on $\mathcal{X}_0^+(p^n)$ corresponding to the double class $1_{I_n^+ a I_n^+}$ where in the inert case

$$a = \begin{pmatrix} p & & \\ & 1 & \\ & & p^{-1} \end{pmatrix}$$

and in the split case,

$$a = \begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad \text{or} \quad a = \begin{pmatrix} p & & \\ & p & \\ & & 1 \end{pmatrix}.$$

We call respectively $F_{p,1}^n$, $F_{v,1}^n$, $F_{v,1}^n F_{v,2}^n$ the corresponding operators.These operators can be defined on the moduli problem $\mathcal{X}_0^+(p^n)^{tor}$ and commutes with their counterparts on $\mathcal{X}_0(p)^{tor} =: \mathcal{X}^{tor}$ (see for example [**PS17**] section 8.2), in the sense that for one of these operators, say g, if we denote the correspondence C and $C_n = C \times_{\mathcal{X}} \mathcal{X}_0^+(p^n)$, with π_g^n and π_g the universal isogeny on C_n and C, we thus have commutatives diagrams,

$$\begin{tikzcd}
 & C_n \arrow[dl, "p_1"'] \arrow[dd, "\pi"] \arrow[dr, "p_2"] & \\
\mathcal{X}_0^+(p^n) \arrow[dd, "\pi"] & & \mathcal{X}_0^+(p^n) \arrow[dd, "\pi"] \\
 & C \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathcal{X} & & \mathcal{X}
\end{tikzcd}$$

and a commutative diagram,

$$\begin{tikzcd}
\mathcal{I}W^+_{\mathcal{X}_0^+(p^n)} \times C_n \arrow[r, "\pi_g^n"] \arrow[d, "p"] & \mathcal{I}W^+_{\mathcal{X}_0^+(p^n)} \times C_n \arrow[d, "p"] \\
\mathcal{I}W^+_{\mathcal{X}} \times C \arrow[r, "\pi_g"] & \mathcal{I}W^+_{\mathcal{X}} \times C
\end{tikzcd}$$

The normalisation of the maps π_g^n and π_g can be done the same way, and we thus deduce that the operators $U_{p,\star}$ in level $\mathcal{X}$ and $U_{p,\star}^n$ in level $\mathcal{X}_0^+(p^n)$ commutes with the pullback by π (i.e. $U_{p,\star}^n(\pi^* f) = \pi^*(U_{p,\star} f)$). Thus, these operators defined on $\omega_w^{\kappa\dagger}$ for any $\kappa \in \mathcal{W}$ w-analytic (with $w \in]n-1, n-\varepsilon_0[$) are the same once we identify (invariant by T_n) sections on some small neighborhood $\mathcal{X}_0^+(p^n)(v)$ of $\omega_w^{\kappa^0\dagger}(-\kappa_{|T_n})$ with sections of $\omega_w^{\kappa\dagger}$ on some small neighborhood $\mathcal{X}(v)$.

In particular to understand the action of $\mathcal{A}(p)$ on the forms corresponding to π^n which appears in $H^0(\mathcal{X}^{tor}(v), \omega_w^{\kappa\varphi\dagger})$ for v small enough, we need to understand the action of $\mathcal{A}_n^+(p)$ on π_p^n.

Definition 10.13. — If π is a representation of $G(\mathbb{Q}_p)$, denote by

$$(\pi^{I_n^+})^{fs} := 1_{I_n^+ a I_n^+}(\pi^{I_n^+}),$$

where a is the diagonal element corresponding to $F_{p,1}^n$ if p is inert, and $(F_{v,1}^n)^2 F_{v,2}^n$ if p is split (in other words a is the double class corresponding to the compact operator U_p in the text, up to twits by a central element). This coincides with the space $V_{A^-}^{K_0}$ of **[Cas95]**, Proposition 4.1.6.

By **[Cas95]** Lemma 4.1.7, this space $(\pi^{I_n^+})^{fs}$ is endowed with an action of $\mathcal{A}_n^+(p)$.

Proposition 10.14. — *Let π be a representation of $G(\mathbb{Q}_p)$. Write $I_n^+ = \overline{N}_n T_n^+ N_n$ its Iwahori decomposition. Then, as $\Sigma = \mathcal{A}_n^+(p)$-module,*

$$(\pi^{I_n^+})^{fs} = (\pi_{N_n})^{T_n^+} \otimes \delta_B^{-1},$$

Proof. — As in [**BC09**] Proposition 6.4.3, this is due to [**Cas95**] Proposition 4.1.4. using the Iwahori decomposition. □

Remark 10.15. — We could also extends a bit the previous isomorphism by adding the action of (the split part of) T/T_n^+ as in [**Cas95**].

Moreover, as π^n is a quotient of an induction (or the induction from a parabolic subgroup in the split case), we will use the same geometric lemma as [**BC09**] Proposition 6.4.4. In particular we only need to calculate the admissible refinement using this lemma, and as this does not assume χ to be unramified, we find exactly the same (automorphic) refinements as if $p \nmid \mathrm{Cond}(\chi)$ in $(((\pi_p^n)^{I_n^+})^{fs})^{ss}$.

Definition 10.16. — Let σ be the refinement corresponding to the one when $p \nmid \mathrm{Cond}(\chi)$ used in [**Her19**] when p is inert (in which case it is unique, see [**Her19**] Proposition 10.7), and to $(\mu|.|^{-1/2})(1,\overline{\chi}(p),p^{-1})$, see [**BC09**] Lemma 8.2.1 when p is split [25]. More precisely, it corresponds to

$$(\mu|.|^{-1/2})(1,\overline{\chi}(p),p^{-1}) : \begin{array}{ccc} T/T_n^+ & \longrightarrow & \mathbb{C}^\times \\ (a,b,c) & \longmapsto & (\mu|.|^{-1/2})(abc)\overline{\chi}(c)|b| \end{array}$$

in the case where p splits, and to

$$(\mu|.|^{-1/2})(1,\overline{\chi}(p),p^{-1}) : \begin{array}{ccc} T/T_n^+ & \longrightarrow & \mathbb{C}^\times \\ (a,e) & \longmapsto & (\mu|.|^{-1/2})(a\overline{a}^{-1}e)\overline{\chi}(e)|a| \end{array}$$

when p is inert. Recall that $T \simeq (\mathcal{O}[1/p])^\times \times (\mathcal{O}[1/p])^1$ in this case.

10.3. Refinements of De Rham representations. — In this subsection, we slightly generalise the well-known notion of refinements (see e.g. [**BC09**] section 2.4) to non-necessarily crystalline representations. This is especially useful for us when $p|\,\mathrm{Cond}(\chi)$.

Definition 10.17. — Let V a n-dimensional, continuous L-representation of G_K, where K is a p-adic field. Assume that V is De Rham, and denote $WD(V)$ the Weil-Deligne representation associated to V (see [**Fon94, BGGT14**]). Assume that L is big enough so that all eigenvalues of the Frobenius φ on $WD(V)$ are defined on L. A ***Refinement*** of V is the datum $(\mathcal{F}_i)_{i=1,\dots,n}$ of a filtration

$$0 \subsetneq \mathcal{F}_1 \subsetneq \cdots \subsetneq \mathcal{F}_n = WD(V),$$

by Weil-Deligne representations.

Just as in the crystalline case, the previous definition more generally applies to a general De Rham (φ,Γ)-module D, to $WD(D)$ (see [**Ber08**]).

Remark 10.18. — Obviously when V is crystalline, this definition coïncides with the one of [**BC09**].

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[25] This refinement is not *ordinary*, in the sense that the normalised Hecke operators F_i won't have slope zero at the corresponding point. In the split case, the other two accessible refinements are also non-ordinary (one of which being even anti-ordinary in the sense of [**BC04**], but unfortunately we can't check crystallinity (i.e. Theorem B.5) at those refinements.

Let D be a De Rham (φ, Γ)-module. Let $(\mathcal{F}_i)$ be a refinement of D, i.e. a filtration of $WD(D)$. Then we can associate to $(\mathcal{F}_i)$ a filtration of D by

$$\mathrm{Fil}_i(D) = (R[1/t]\mathcal{F}_i) \cap D.$$

This filtration is saturated, and thus defines a triangulation of D (see [**BC09**], section 2.3).

Proposition 10.19. — *The previous map $(\mathcal{F}_i) \mapsto (\mathrm{Fil}_i\, D)$ induces a bijection between the set of refinements of D and the set of triangulations of D.*

Proof. — This is [**Ber08**] Théorème A and Corollaire III.2.5. □

In the particular case of an automorphic representation π of our unitary group G, with associated Galois representation ρ_π (for example $\rho = 1 \oplus \chi_p^c \oplus \varepsilon$ associated to the automorphic representation π^n of the previous subsection), we have distinguished – we call them *accessible*, (galois) refinements for $\rho_{\pi,v}$ which correspond to the (automorphic) refinements for the action of $\mathcal{A}_n^+(p)$ on π_v^n (for $v|p$). Such refinements exist only if $(\pi_v)^{I_n^+} \neq 0$ for some n. The association is explained in [**BC09**] (when G is split at v) for unramified representations, and for $U(3)(\mathbb{Q}_p)$ (when v is inert) in [**Her19**] section 10.5. This can be generalized for non-necessarily unramified π_v, verbatim when there is no monodromy. For example, to the refinement σ of definition 10.16, is associated the following refinement of $\rho = 1 \oplus \chi_p^c \oplus \varepsilon$:

$$\left\{ \begin{array}{ll} 0 \subsetneq LL(1) \subsetneq LL(1) \oplus LL(\chi_p^c) \subsetneq WD(\rho_{G_p}) & \text{when } p \text{ is inert.} \\ 0 \subsetneq LL(\chi_v^c) \subsetneq LL(1) \oplus LL(\chi_v^c) \subsetneq WD(\rho_{G_v}) & \text{when } p \text{ is split.} \end{array} \right. \tag{5}$$

Here 1 is the trivial representation of $E_v^\times$, and LL denotes the Local-Langlands correspondence.

10.4. Constructing the extension. — We thus take a prime p unramified in E, which can be 2 or not, and which can divide $\mathrm{Cond}(\chi)$ or not. Let $\mathcal{E}$ be the eigenvariety of level $N = \mathrm{Cond}(\chi)^p$ (the prime-to-p-part of the conductor) associated to $(G)U(2,1)_E$ and p by Theorem 9.5. It is equipped with a map $w : \mathcal{E} \longrightarrow \mathcal{W}$, and there is a point $y \in \mathcal{E}$ which coincides with the representation π^n together with its refinement σ by definition 10.16 and proposition 10.10. For all $\mathcal{Z} \subset \mathcal{E}$, we have associated to the automorphic form corresponding to z a Galois representation

$$\rho_z : G_E \longrightarrow \mathrm{GL}_3(\overline{\mathbb{Q}_p}),$$

which is moreover polarised in the following way :

$$\rho_z^\perp \simeq \rho_z(-1) := \rho_z \varepsilon^{-1},$$

where ε denote the cyclotomic character. Let us be more precise : we will change a bit the convention used in [**Her19**] to stick with the one of [**BC09**] (this will make things easier to treat the case $p|\,\mathrm{Cond}(\chi)$). Denote for an automorphic representation π of $U(2,1)$ of regular weight ρ'_π the associated p-adic Galois representation by [**BC09**] Conjecture 6.8.1, which is know to exists, see Remark 6.8.3, (vi) of [**BC09**]. For $z \in \mathcal{Z}$ associated to a modular form f_z, denote by Π any irreducible constituent of the representation of (the restriction to) $U(2,1)(\mathbb{A})$ generated by f_z. Then we set

$$\rho_z = \rho'_\pi \nu,$$

where ν is defined in **[BC09]** Lemma 8.2.3, and is associated by class field theory to $\mu^{-1}|.|^{3/2}$[26]. In particular it satisfies $\nu^\perp = \nu(-3)$. Thus $\rho_z^\perp = \rho_z(-1)$. Moreover for $z \in \mathcal{Z}$ of classical (automorphic) weight $(k_1 \geqslant k_2, k_3)$, the Hodge-Tate weights of ρ_z are given by[27]

$$((-k_1, 1-k_2, k_3-1), (-k_3, k_2-2, k_1-1)) = \begin{cases} (\mathrm{HT}_\tau, \mathrm{HT}_{\overline{\tau}})(\rho_z) & \text{if } p \text{ is inert} \\ (\mathrm{HT}_v, \mathrm{HT}_{\overline{v}})(\rho_z) & \text{if } p \text{ splits} \end{cases}$$

Proposition 10.20. — *There exists a pseudo character on $\mathcal{E}$,*

$$T : G_E \longrightarrow \mathcal{O}(\mathcal{E}),$$

such that for all $z \in \mathcal{Z}$, T_z is the trace of ρ_z. Moreover, $T^\perp = T(-1)$.

Proof. — This is **[Che04]** Proposition 7.1. □

We need a particular point on $\mathcal{E}$.

Proposition 10.21. — *Suppose* $\mathrm{ord}_{s=0} L(\chi, s)$ *is even and $L(\chi, 0) = 0$. There exists a point $y \in \mathcal{E}$ corresponding to a (non-tempered) automorphic representation π^n. The point y is non-classical if $p|$* $\mathrm{Cond}(\chi)$[28] *but is classical otherwise. Moreover its p-adic weight $w(\chi)$ is of the form $w(\chi)^{alg} w(\chi)^{sm}$ where $w(\chi)^{sm}$ is the smooth (finite order) character of Proposition 10.10, and $w(\chi)^{alg}$ is the algebraic character*

$$\begin{array}{ccc} \mathcal{O}^\times \times \mathcal{O}^1 & \longrightarrow & \overline{\mathbb{Q}_p}^\times \\ (x,y) & \mapsto & \tau(x)\tau(y)^{2-a}\sigma\tau(x) \end{array} \quad \textit{if } p \textit{ is inert}$$

$$\begin{array}{ccc} \mathbb{Z}_p^3 & \longrightarrow & \overline{\mathbb{Q}_p}^\times \\ (x,y,z) & \longmapsto & xy^{2-a}z \end{array} \quad \textit{if } p \textit{ is split}$$

At the point y, the evaluation T_y is given by the trace of $1 \oplus \varepsilon \oplus \chi^c$ and the refinement is given by σ of definition 10.16, i.e. it is the refinement (5).

Proof. — This is a translation of Proposition 10.10 with the normalisation of T. □

We freely use the notation of **[KPX14]** concerning φ, Γ-modules. Denote δ_i for $i = 1, 2, 3$ the character,

$$\delta_i : K^\times \longrightarrow \mathcal{O}(\mathcal{E})^\times,$$

such that $\delta_i(p) = F_i$[29] and, in the inert case, recall that we have on $\mathcal{W}$ two universal morphisms,

$$\kappa_1 : x \in \mathcal{O}^\times \longmapsto \kappa_1(x) \in \mathcal{O}(\mathcal{W})^\times, \quad \text{and} \quad \kappa_2 : y \in \mathcal{O}^1 \longmapsto \kappa_2(y) \in \mathcal{O}(\mathcal{W})^\times,$$

such that at classical points $\kappa = (k_1, k_2, k_3) \in \mathbb{Z}^3$, we have

$$\kappa_{1|\kappa}(x) = \tau(x)^{k_1}\overline{\tau}(x)^{k_3} \quad \text{and} \quad \kappa_{2|\kappa}(y) = \tau(y)^{k_2}.$$

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(26) Careful to the normalisation of the Local Langlands correspondence in **[BC09]**

(27) We choose the normalisation of the Hodge-Tate weight such that the cyclotomic character has Hodge-Tate weight -1, as in **[BC09]**

(28) More precisely, it is non classical without level at p as its system of Hecke eigenvalues doesn't appear in $H^0(\mathcal{X}, \omega^\kappa)$, but appears in $H^0(\mathcal{X}_0^+(p^n), \omega^\kappa(\varphi))$.

(29) These $F_i \in \mathcal{O}(\mathcal{E})^\times$ already appeared in proof of proposition 10.1. These are the functions given by a basis of the Hecke operator in $\mathcal{A}(p)$.

We set

$$\delta_{1|\mathcal{O}_K^\times} = (\kappa_1)x_{\overline{\tau}}^{-2},$$

$$\delta_{2|\mathcal{O}_K^\times} : y \in \mathcal{O}_K^\times \longmapsto \kappa_2(y/\overline{y})\tau(y)^{-1},$$

$$\delta_{3|\mathcal{O}_K^\times} = (\kappa_1^c)^{-1}x_\tau x_{\overline{\tau}}^{-1} = (\delta_{1|\mathcal{O}_K^\times}^c)^{-1}x_\tau^{-1}x_{\overline{\tau}}^{-1}.$$

In particular we have $\delta_3 = \overline{\delta_1}^{-1}x$. In the split case, we set

$$\delta_i : \mathbb{Q}_p^\times \longrightarrow \mathcal{O}(\mathcal{E})^\times,$$

with $\delta_i(p) = F_i$ and as we have universal characters on $\mathcal{W}$,

$$\kappa_i : \mathbb{Z}_p^\times \longrightarrow \mathcal{O}(\mathcal{W})^\times,$$

such that for classical weights $(k_1, k_2, k_3) \in \mathbb{Z}^3$,

$$\kappa_1(x) = x^{k_1}, \quad \kappa_2(x) = x^{k_2}, \quad \kappa_3(x) = x^{k_3},$$

We set

$$\delta_{1|\mathbb{Z}_p^\times} = \kappa_1, \quad \delta_{2|\mathbb{Z}_p^\times} = \kappa_2 x^{-1}, \quad \delta_{3|\mathbb{Z}_p^\times} = x\kappa_3^{-1}.$$

In particular, we can define $\mathrm{wt}_i := -\mathrm{wt}(\delta_i) \in \mathcal{O}(\mathcal{W})^\Sigma$ the opposite of the derivative at 1 of δ_i (see [**KPX14**] Definition 6.1.6). In particular $\mathcal{E}, \mathcal{Z}$ and the functions δ_i satisfies the hypothesis of Corollary 6.3.10 of [**KPX14**] (excepts possibly the irreducibility condition).

Denote by $A = \mathcal{O}_y$ the rigid analytic local ring of $\mathcal{E}$ at y, and K its total fraction ring. The pseudo character T on $\mathcal{E}$ induces one on A, and denote by $I_{tot} \subset A$ its total reducibility ideal (see [**BC09**] Proposition 1.5.1, Definition 1.5.2.) In particular for any $J \supset I_{tot}$ on A/J we can write

$$T \otimes A/J = T_1 + T_{\overline{\chi}} + T_\varepsilon.$$

Proposition 10.22. — *The reducibilty locus* $\mathrm{Spec}(A/I_{tot})$ *is a proper closed sub-scheme of* $\mathrm{Spec}(A)$*, i.e.* $I_{tot} \neq \{0\}$*. More precisely, if* p *is inert we have that*

$$\mathrm{wt}_\tau(\delta_1) - \mathrm{wt}_\tau(\delta_3) \equiv \mathrm{wt}_\tau(\delta_1)(y) - \mathrm{wt}_\tau(\delta_3)(y) \pmod{I_{tot}},$$

and

$$\mathrm{wt}_{\overline{\tau}}(\delta_1) - \mathrm{wt}_{\overline{\tau}}(\delta_3) \equiv \mathrm{wt}_{\overline{\tau}}(\delta_1)(y) - \mathrm{wt}_{\overline{\tau}}(\delta_3)(y) \pmod{I_{tot}},$$

and similarly (with $\tau, \overline{\tau}$ *changed by* $v, \overline{v}$*) is* p *splits.*

Proof. — Let $I \supset I_{tot}$ a finite length ideal of $\mathcal{O}_{y_0} = A$. We thus have for $j = \{1, \overline{\chi}, \varepsilon\}$,

$$T_j : G_{E,S} \longrightarrow A/I,$$

a (continuous) character, such that $T_j \pmod{\mathfrak{m}_A} = j$.

As T_j is a character, by [**KPX14**] Theorem 6.2.14, there exists a character

$$\delta_j' : K^\times \longrightarrow (A/I)^\times,$$

such that the φ, Γ-module associated to T_j, $D_{rig}(T_j)$ is isomorphic to $R_{A/I}(\pi_K)(\delta_j')$. From now on we just write this last space $R(\delta_j')$. We will determine δ_j'. Recall that $j \in \{1, \overline{\chi}, \varepsilon\}$

and $i \in \{1,2,3\}$. We choose the bijection between these two spaces, which corresponds to the refinement 10.16, more precisely,

$$\begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & \overline{\chi} \\ 3 & \mapsto & \varepsilon \end{array}$$

Thus $T_2 := T_{\overline{\chi}}$ and $T_3 := T_\varepsilon$, for example. By Lemma B.3, we have in particular a map

$$R(\delta_i) \hookrightarrow D_{rig}(T_i) \simeq R(\delta_i').$$

To determine δ_i' the character of T_i, we still need to know the weight of T_i. We know by Lemma B.2 that T_i has its Sen operator killed by

$$\prod_{i=1}^{3}(T - \mathrm{wt}_i) \in A/I[T]^\Sigma.$$

Moreover, at $y \in \mathcal{E}$, we have, if p splits,

$$(\mathrm{wt}_1^v, \mathrm{wt}_2^v, \mathrm{wt}_3^v) = (-1, a-1, 0) \quad \text{and} \quad (\mathrm{wt}_1^{\overline{v}}, \mathrm{wt}_2^{\overline{v}}, \mathrm{wt}_3^{\overline{v}}) = (-1, -a, 0)$$

and if p is inert,

$$(\mathrm{wt}_1^\tau, \mathrm{wt}_2^\tau, \mathrm{wt}_3^\tau) = (-1, a-1, 0) \quad \text{and} \quad (\mathrm{wt}_1^{\overline{\tau}}, \mathrm{wt}_2^{\overline{\tau}}, \mathrm{wt}_3^{\overline{\tau}}) = (-1, -a, 0).$$

Thus, if $a \geqslant 2$ these weights are distincts at y. Thus we can calculate the Hodge-Tate-Sen weight of T_i : T_1 has weight wt_3, $T_{\overline{\chi}}$ has weight wt_2 and T_ε has weight wt_1. Similarly at v and $\overline{v}$ if p splits. If $a = 1$, we can't a priori distinguish the two weights $\mathrm{wt}_2^v, \mathrm{wt}_3^v$ at v and $\mathrm{wt}_1^{\overline{v}}, \mathrm{wt}_2^{\overline{v}}$ at $\overline{v}$ (similarly at τ and $\overline{\tau}$), but we know that $T_\varepsilon = T_3$ has weight $\mathrm{wt}(\delta_1)$ at v, and that T_1 has weight $\mathrm{wt}(\delta_3)$ at $\overline{v}$.

Suppose p is split, using Lemma B.3 and Lemma B.4 for T_ε, we have (evaluating at y to have the value of t_σ, k_σ),

$$\mathrm{wt}_v(\delta_1) - \mathrm{wt}_v(\delta_3) - (\mathrm{wt}_v(\delta_1)(y) - \mathrm{wt}_v(\delta_3)(y)) \in I.$$

Using that $\delta_3 = \overline{\delta_1}^{-1}x$ (or using lemma B.4 for T_1 at $\overline{v}$), we get the result for $\overline{v}$. This is identical if p is inert. $\square$

We also need the following result, which is a corollary of theorem B.5.

Corollary 10.23. — *$Ext_T(1,i) \subset H^1_f(E,i)$, for $i = \overline{\chi}$ or ε.*

Proof. — Indeed, the theorem B.5 gives that any extension in $\mathrm{Ext}_T(1,i)$ is crystalline at all place above p (as the Frobenius eigenvalues of i are different from 1). At v a place dividing $\ell \neq p$, if $v \nmid \mathrm{Cond}(\chi)$, by hypothesis on the level of $\mathcal{E}$, the dense set of classical points Z are unramified at v, thus $T(I_v) = 1$ on $\mathcal{E}$ (as $\mathcal{E}$ is reduced) and thus $\mathrm{Ext}_T(1,i)$ consists of unramified extensions at v.

Now suppose $v | \mathrm{Cond}(\chi)$. If $i = \varepsilon$, any extension is automatically unramified. Suppose $i = \overline{\chi}$. By choice of the type J outside p on $\mathcal{E}$, we know ([**BC04**] Proposition 4.2 or [**Her19**] proposition 10.21) that for all $z \in Z$, there exists a subgroup $I' \subset I_v$ such that $\rho_z(I') = \{1\}$. Thus, $T(I') = 1$ and for all $x \in \mathcal{E}, \rho_x(I') = 1$. Thus, T_{I_v} is locally constant, and the same for $\rho_{x|I_v}$ (as it is semi-simple as I' acts trivially). Up to extending scalars, evaluating at ρ_y, we get

$$T_{|I_v} = (1 \oplus 1 \oplus \overline{\chi}_{|I_v}) \otimes \mathcal{O}_U,$$

for some neighborhood U of x. But as we have a morphism

$$M_1/IM_1 \oplus T_i \longrightarrow \rho_c \longrightarrow 0,$$

we have that $\rho_c(I') = 1$, thus ρ_c is semi-simple, thus $\rho_{c|G_v} \in H^1_f(G_v, \overline{\chi})$. □

We have the following improvement of Theorem 10.3 :

***Theorem 10.24.** — Let χ be a polarized algebraic Hecke character as in Theorem 10.3. Suppose that $L(\chi, s)$ vanishes with even (non-zero) order at $s = 0$. Let p be unramified in E. Then*

$$H^1_f(E, \chi_p) \neq \{0\}.$$

Proof. — Let $e_1, e_{\overline{\chi}}, e_\varepsilon$ be the indempotents as in Appendix B, and denote $A_{i,j}$, for $i, j \in \{1, \overline{\chi}, \varepsilon\}$ the corresponding A-modules. Then as in **[BC09]** Lemma 8.3.2, we get

$$I_{tot} = A_{1,\overline{\chi}} A_{\overline{\chi},1}.$$

But if $Ext_T(1, \overline{\chi}) = 0$, then $A_{\overline{\chi},1} = A_{\overline{\chi},\varepsilon} A_{\varepsilon,1}$ (**[BC09]**, Theorem 1.5.5). Thus $A_{\overline{\chi},1} A_{1,\overline{\chi}} = A_{\overline{\chi},\varepsilon} A_{\varepsilon,1} A_{1,\overline{\chi}}$. But as $H^1_f(E, \varepsilon) = \{0\}$, we get by the same reasoning

$$A_{\varepsilon,1} = A_{\varepsilon,\overline{\chi}} A_{\overline{\chi},1}.$$

Thus,

$$I_{tot} = A_{\overline{\chi},\varepsilon} A_{\varepsilon,\overline{\chi}} A_{\overline{\chi},1} A_{1,\overline{\chi}} \subset \mathfrak{m} A_{\overline{\chi},1} A_{1,\overline{\chi}} = \mathfrak{m} I_{tot}.$$

Thus $I_{tot} = 0$, contradicting proposition 10.22. □

Appendix A. Cohomology of cuspidal automorphic sheaves

***Proposition A.1.** — (Lan, **[Lan17]** Theorem 6.1) Let $\mathfrak{X}_1(p^n)^*$ the minimal compactification of $\mathfrak{X}_1(p^n)$, defined by normalisation of the minimal compactification with our fixed auxiliary level, as in **[Lan16a]**, Proposition 6.1. There is a proper surjection $p : \mathfrak{X}_1(p^n)^{tor} \longrightarrow \mathfrak{X}_1(p^n)^*$.*

Definition A.2. — The (μ-ordinary) Hasse invariant ${}^\mu\,\mathrm{Ha}$ descends to $\mathfrak{X}_1(p^n)^*$ (modulo p), and we can thus define $\mathfrak{X}_1(p^n)^{\mu-full*}(v)$ to be the normalisation in its generic fiber of the greatest open in the blow up of $({}^\mu\mathrm{Ha}, p^v)$ where this ideal is generated by ${}^\mu\,\mathrm{Ha}$. Its generic fiber is $\mathcal{X}_1(p^n)^{\mu-full*}(v)$, a strict neighborhood of the (full) μ-ordinary locus. Denote $\mathcal{X}_1(p^n)^*(v)$ the (union of) connected components which contains a point of maximal degree, and as $\mathfrak{X}_1(p^n)^{\mu-full*}$ is normal in its generic fiber, there is an associated open $\mathfrak{X}_1(p^n)^*(v)$. We thus have a map,

$$\pi(v) : \mathfrak{X}_1(p^n)^{tor}(v) \longrightarrow \mathfrak{X}_1(p^n)^*(v).$$

For all this section, except the last two results (Corollary A.5, Theorem A.6), we forgot the notation concerning the level at p, and denote $\mathfrak{X}_1(p^n)^{tor}(v)$ by $\mathfrak{X}^{tor}(v)$, and similarly for $\mathfrak{X}(v), \mathfrak{X}^*(v), \mathfrak{X}^{tor}, \mathfrak{X}, \mathfrak{X}^*$. We thus have the previous map,

$$\pi(v) : \mathfrak{X}^{tor}(v) \longrightarrow \mathfrak{X}^*(v).$$

Assume that our fan Σ is smooth and projective. We have the following vanishing result.

Proposition A.3. *— Denote by $D(v)$ the boundary in $\mathfrak{X}^{tor}(v)$. Then, for all $q > 0$,*

$$R^q\pi(v)_*\mathcal{O}(-D(v)) = 0.$$

Proof. — This is essentially Lan's result (see [**Lan17**] Proposition 8.6), slightly modified because of the neighborhood we chose. First, note that we can prove it for $\mathfrak{X}^{\mu-full}(v)^{tor}$ and $\mathfrak{X}^{\mu-full}(v)^*$ and then localise (as the schemes are normal and thus have same connected component as their rigid fiber) to $\mathfrak{X}^{tor}(v)$ and $\mathfrak{X}^*(v)$. From now on and until the end of this proof, we denote $\mathfrak{X}^?(v)$ the neighborhood of the full μ-ordinary locus in $\mathfrak{X}^?$. By the formal functions theorem we can work on formal completions of geometric points $\overline{x} \in \mathfrak{X}^*(v)$, and we need to prove that $H^q(\mathfrak{X}^{tor}(v)^{\wedge}_{\overline{x}}, \mathcal{O}(-D(v))) = 0$ for all $q > 0$. Let us describe the completions at $\overline{x}$ of $\mathfrak{X}(v)^{tor}$. Let $\mathfrak{Z}$ be a stratum of $\mathfrak{X}^*$ ([**Lan16a**] Theorem 12.1, it depends on a choice of a cusp datum), and denote $\mathfrak{Z}^*(v)$ be the base change of $\mathfrak{Z}$ to $\mathfrak{X}^*(v)$, then $\mathfrak{Z}^*(v)$ is locally closed in $\mathfrak{X}^*(v)$. In [**Lan16a**, Theorem 10.13] (see also notations of [**Lan16b**, Sect. 4], and [**Lan17**, Theorem 6.1]), local charts for $\mathfrak{X}^{tor}$ over $\mathfrak{X}^*$ are constructed using normalization of local charts in an auxiliary Shimura datum. They have the following shape

$$\begin{tikzcd} & & \Xi \arrow[dl] \arrow[dd, "i"] \\ \mathfrak{Z} & C \arrow[l] & \\ & & \Xi(\sigma) \arrow[ul] \end{tikzcd}$$

with $i : \Xi \hookrightarrow \Xi(\sigma)$ an affine torus embedding, and if $\mathfrak{U}_\sigma$ denote the completion of $\Xi(\sigma)$ along its closed strata, $\{\mathfrak{U}_\sigma\}$ glue together to a formal scheme $\mathfrak{X} = \mathfrak{X}_\Sigma$, and $(\mathfrak{X}^{tor})^{\wedge}_{\mathfrak{Z}} \simeq \mathfrak{X}/\Gamma$ with Γ acting on $\mathfrak{X}$ freely and

$$\mathfrak{X} \longrightarrow \mathfrak{X}/\Gamma \simeq (\mathfrak{X}^{tor})^{\wedge}_{\mathfrak{Z}} \tag{6}$$

is a local isomorphism. All the maps described before are flat. Denote $\mathfrak{Z}(v)$ be the normalisation in its rigid fiber of the open $\mathfrak{Z}(v)^0$ in the blow-up of the ideal $I = ({}^\mu\,\mathrm{Ha}, p^v)$ in $\mathfrak{Z}$ where I is generated by ${}^\mu\,\mathrm{Ha}$. It is not a priori equal to $\mathfrak{Z}^*(v)$ (which is defined by base change). Let $C \longrightarrow \mathfrak{Z}$ be the proper scheme, normal over $\mathcal{O}$ ([**Lan16a**] Proposition 8.4), but as it is constructed using normalisation of $C_{aux} \longrightarrow \mathfrak{Z}_{aux}$ for an auxiliary datum, where $C_{aux} \longrightarrow \mathfrak{Z}_{aux}$ is an abelian scheme torsor over a finite etale formal scheme above $\mathfrak{Z}_{aux}$, thus is smooth, and as normalisation commutes with smooth base change ([**Sta18**, Tag 03GV]) we have $C = C_{aux} \times_{\mathfrak{Z}^{aux}} \mathfrak{Z}$, and $C \longrightarrow \mathfrak{Z}$ is smooth again. Then we define $C(v)$ as the normalisation in its rigid fiber of $C(v)^0$, the open in the Blow-up of $I = ({}^\mu\,\mathrm{Ha}, p^v)$ where I is generated by ${}^\mu\,\mathrm{Ha}$. Then, as the Blow-up commutes with flat base change, we have $C(v)^0 = C \times_{\mathfrak{Z}} \mathfrak{Z}(v)^0$ and $C(v)^0 \longrightarrow \mathfrak{Z}(v)^0$ is smooth thus again $C(v) = C \times_{\mathfrak{Z}} \mathfrak{Z}(v)$. Define analogoulsy the local models (see [**Lan16b**] section 4.) $\mathfrak{U}_\sigma(v)$ and $\mathfrak{X}(v)$ (as the fan Σ is smooth, normalisation commutes with base change). As all operations commutes, we have shown that

$$\mathfrak{X}(v)/\Gamma \simeq (\mathfrak{X}^{tor})^{\wedge}_{\mathfrak{Z}}(v).$$

We can describe locally $\mathfrak{X}^{tor}$ over $\mathfrak{X}^*$ by $\mathfrak{X}$ (by equation 6, see [**Lan16a**] Theorem 10.3, [**Lan17**] Theorem 6.1 (4)) and also for $\mathcal{X}^{tor}(v)$ over $\mathcal{X}^*(v)$, i.e. in rigid fiber, as this is just localisation over an open subset. Denote by $\mathfrak{X}^*(v)^0$ denote the open of the blow-up where the ideal is generated by ${}^\mu\,\mathrm{Ha}$ (i.e. before taking the normalisation in its rigid fiber), and similarly for $\mathfrak{X}^{tor}(v)^0$. Then $\mathfrak{X}^{tor} \longrightarrow \mathfrak{X}^*$ is not flat à priori, but as $({}^\mu\,\mathrm{Ha}, p^v)$ is in both cases a regular sequence, this implies that the admissible formal blow-up in both cases is given by the closed subset of equation $(X{}^\mu\,\mathrm{Ha} - Yp^v)$ in $Proj(\mathcal{O}_{\mathfrak{X}^?}[X,Y])$ (see e.g. [**Bos14**], Proposition 7. (iii)). Thus this admissible blow-up commutes with the base change $\mathfrak{X}^{tor} \longrightarrow \mathfrak{X}^*$. In particular, $\mathfrak{X}^{tor}(v)^0 = \mathfrak{X}^{tor} \times_{\mathfrak{X}^*} \mathfrak{X}^*(v)^0$. Thus $\mathfrak{X}^{tor}(v)$ is the normalisation of $\mathfrak{X}^{tor} \times_{\mathfrak{X}^*} \mathfrak{X}^*(v)^0$ in its rigid fiber and we have a map

$$\mathfrak{X}^{tor}(v) \longrightarrow \mathfrak{X}^*(v).$$

Denote $\mathfrak{Z}^{tor}(v) = \mathfrak{Z} \times_{\mathfrak{X}^{tor}} \mathfrak{X}^{tor}(v)$, this is a locally closed (formal-)subscheme of $\mathfrak{X}^{tor}(v)$ and coïncides with the pullback of $\mathfrak{Z}^*(v)$ through the previous map. We claim that

$$(\mathfrak{X}^{tor}(v))^\wedge_{\mathfrak{Z}^*(v)} = (\mathfrak{X}^{tor}(v))^\wedge_{\mathfrak{Z}^{tor}(v)} \simeq (\mathfrak{X}^{tor})^\wedge_{\mathfrak{Z}}(v) \simeq \mathfrak{X}(v)/\Gamma,$$

so that $\mathfrak{X}^{tor}(v)$ over $\mathfrak{X}^*(v)$ is correctly described by $\mathfrak{X}(v)$. We only need to prove the first isomorphism. Denote (abusively) $I = ({}^\mu\,\mathrm{Ha}, p^v)$ the ideal on the various formal schemes, and $(\cdot)^\wedge_{\mathfrak{Z}}$ the completion along $\mathfrak{Z}$ or its pullback in those schemes (in particular this is the completion along $\mathfrak{Z}^{tor}(v)$ for $\mathfrak{X}^{tor}(v)$). As $\mathfrak{X}^{tor}$ is noetherian, $(\mathfrak{X}^{tor})^\wedge_{\mathfrak{Z}} \longrightarrow \mathfrak{X}^{tor}$ is flat, and as blow-up commutes with flat base change, we have

$$(\mathfrak{B}l_I(\mathfrak{X}^{tor}))^\wedge_{\mathfrak{Z}} = \mathfrak{B}l_I((\mathfrak{X}^{tor})^\wedge_{\mathfrak{Z}}),$$

and $(\mathfrak{X}^{tor,\wedge}_{\mathfrak{Z}})(v)$ is an open in the normalisation of $\mathfrak{B}l_I((\mathfrak{X}^{tor})^\wedge_{\mathfrak{Z}}) = (\mathfrak{B}l_I(\mathfrak{X}^{tor}))^\wedge_{\mathfrak{Z}}$. But $\mathfrak{X}^{tor}$, and thus $\mathfrak{B}l_I(\mathfrak{X}^{tor}))$ is quasi-excellent, normalisation and $\mathfrak{Z}$-adic completion commutes.(30) Thus $\mathfrak{X}^{tor,\wedge}_{\mathfrak{Z}}(v)$ is the $\mathfrak{Z}$-adic completion of the open in the normalization of $\mathfrak{B}l_I(\mathfrak{X}^{tor}))$, i.e. of $\mathfrak{X}^{tor}(v)$, thus this is $\mathfrak{X}^{tor}(v)^\wedge_{\mathfrak{Z}^{tor}(v)}$. The etale, local isomorphism

$$\mathfrak{X} \longrightarrow (\mathfrak{X}^{tor})^\wedge_{\mathfrak{Z}},$$

can thus be seen over $\mathfrak{X}^*(v)$, and we get that

$$(\mathfrak{X}^{tor}(v))^\wedge_{\mathfrak{Z}^*(v)} \simeq \mathfrak{X}(v)/\Gamma.$$

Now if $\overline{x}$ is a geometric point of $\mathfrak{X}^*(v)$, lying over $\mathfrak{Z}^*(v)$, we deduce that

$$\mathfrak{X}^{tor}(v)^\wedge_{\overline{x}} \simeq \mathfrak{X}^\wedge_{\overline{x}}/\Gamma,$$

Then according to [**Lan16b**] Theorem 3.9 (and especially section 7), and [**Lan17**] Theorem 8.6 it is sufficient to prove the analog of Proposition 8.3 (of [**Lan17**]) for $p(v) : C(v) \longrightarrow \mathfrak{Z}(v)$. But $p(v)$ is also proper (it is a base change), and the pullback of the sheaf $\Psi(\ell)$ is relatively ample over $\mathfrak{Z}(v)$, thus the same proof applies. $\square$

In the following, we denote for a object X over $\mathrm{Spec}(\mathcal{O})$ or $\mathrm{Spf}(\mathcal{O})$ and $n \in \mathbb{N}^*$, X_n the base change to $\mathrm{Spec}(\mathcal{O}/p^n)$. We also denote, as in [**AIP15**], $\mathfrak{W}(w)^0$ the analogous weight space, but forgetting the torsion part when constructing $\mathfrak{W}(w)$. This can be seen for example as characters in $\mathfrak{W}(w)$ being trivial on the torsion part of $T(\mathbb{Z}_p)$, but we don't fix such an identification.

Proposition A.4. — *Consider the following diagram, for $m \geqslant n$,*

(30) See [**Gro67**, IV$_2$ 7.8.3(v), and proof of 7.6.1]

$$\begin{tikzcd} \mathfrak{X}^{tor}(v)_m \arrow[r, "i"] \arrow[d, "\pi_m"] & \mathfrak{X}^{tor}(v)_n \arrow[d, "\pi_n"] \\ \mathfrak{X}^*(v)_m \arrow[r, "i'"] & \mathfrak{X}^*(v)_n \end{tikzcd}$$

We have the equality,

$$i'*\pi_{n*}\mathfrak{w}_{w,n}^{\kappa^0\dagger}(-D) = \pi_{m,*}i^*\mathfrak{w}_{w,m}^{\kappa^0\dagger}(-D).$$

In particular, $\pi_*\mathfrak{w}_w^{\kappa^0\dagger}(-D)$ is a small formal Banach sheaf on $\mathfrak{X}^*(v) = \mathfrak{X}_1(p^n)^*(v)$. Similarly for $(\pi\times 1)_*\mathfrak{w}_w^{\kappa^{0,univ}\dagger}(-D)$ on $\mathfrak{X}^*(v) \times \mathfrak{W}(w)^0$. Moreover $H^i(\mathfrak{X}^*(v), \pi_*\mathfrak{w}_w^{\kappa^0\dagger}(-D))[1/p]$ vanishes for $i \geqslant 1$ (similarly for the higher direct image of $(\pi \times 1)_*\omega_w^{\kappa^{0,univ}\dagger}(-D)$ on $\mathcal{W}(w)$).

Proof. — The proof is the same as in [**AIP15**] or [**Bra16**], except that we stay at level $\mathfrak{X}_1(p^n)(v)$ (which is easier), as the map $\mathfrak{X}_1(p^n)(v) \longrightarrow \mathfrak{X}(v)$ is not finite in our situation. We can prove as in [**AIP15**] that $\mathfrak{w}_w^{\kappa^0\dagger}(-D)$ is a direct limit of sheaves whose cokernel is a successive extension of the sheaf $\mathcal{O}_{\mathfrak{X}_1(p^n)^{tor}(v)}(-D)$. Thus, it is enough to show that

$$R^1\pi_*\mathcal{O}_{\mathfrak{X}_1(p^n)^{tor}(v)}(-D) = 0,$$

but this is the previous proposition. This implies also that $R^i\pi_*\mathfrak{w}_w^{\kappa^0\dagger}(-D) = 0$ for $i > 0$. Moreover, as $\pi_*\mathfrak{w}_w^{\kappa^0\dagger}(-D)$ is small on $\mathfrak{X}_1(p^n)^*(v)$ which is generically affinoid, Theorem A.1.2.2 of [**AIP15**] implies its higher cohomology vanishes after inverting p. □

Exactly as in [**AIP15**], section 8.2, we deduce the following two results. Now we go back to the notation $\mathfrak{X}_1(p^n)^{tor}(v)$ to denote the (integral toroïdal compactification of the) Shimura variety with level "$\Gamma_1(p^n)$" at p, and $\mathcal{X}^{tor}(v)$ denote the rigid analog, with Iwahori level at p, as in the rest of the text.

Corollary A.5. — *The module*

$$M_{v,w,cusp}^{0,un} := H^0(\mathfrak{X}_1(p^n)^{tor}(v) \times \mathfrak{W}(w)^0, \mathfrak{w}_w^{\kappa^{0,un}\dagger}(-D))[1/p]$$

is a projective $\mathcal{O}_{\mathfrak{W}(w)^0}[1/p]$-module, and for all $\kappa \in \mathcal{W}(w)^0$, the specialisation

$$M_{v,w,cusp}^{0,un} \longrightarrow H^0(\mathfrak{X}_1(p^n)^*(v), \mathfrak{w}_w^{\kappa^0\dagger}(-D))[1/p],$$

is surjective.

Theorem A.6. — *For all v, w the module $M_{v,w}^{un} := H^0(\mathcal{X}^{tor}(v) \times \mathcal{W}(w), \omega_w^{\kappa^{un}\dagger}(-D))$ is a projective $\mathcal{O}_{\mathcal{W}(w)}$-module, and for all $\kappa \in \mathcal{W}(w)$, the specialisation map*

$$M_{v,w,cusp}^{un} \longrightarrow M_{v,w,cusp}^{\kappa},$$

is surjective. Moreover $H^i(\mathcal{X}^{tor}(v) \times \mathcal{W}(w), \omega_w^{\kappa^{un}\dagger}(-D))$ vanishes for $i > 0$.

Appendix B. Families and triangulations

In this appendix we generalise the tools used in [**BC09**] to prove the theorem in section 10. Fortunately, this is mainly a matter of reformulation, as most of the work is done in [**KPX14**]. From now on, we take $\mathcal{E}$ to be the eigenvariety for $(G)U(2,1)_{E/\mathbb{Q}}$ and p a prime unramified in E, constructed in section 9 (see also [**Bra16**] for p split in E and [**Her19**] for p inert not equal to 2), which is 3-dimensional or its variant with weight $k_2 \in \mathbb{Z}$ fixed, which coincide with the base change by

$$\mathcal{W}_{\mathcal{O}^\times} \hookrightarrow \mathcal{W},$$

which is 2-dimensional. Automorphically, the second construction "fixes the central character" (which can "move" in the three dimensional eigenvariety, but keeping its polarisation ; in particular even in the 3-dimensional eigenvariety, we can't twist automorphic forms by a power of the norm character). In any case we always have $Z \subset \mathcal{E}$ a strongly Zariski-dense subset consisting of classical automorphic forms of integral (= algebraic) weight. This space is not dense for the analytic topology, as it is already the case in $\mathcal{W}$. We can define Z^{la} the subset of $\mathcal{E}$ of classical automorphic forms possibly with level at p, and locally algebraic weight-character $\kappa \in \mathcal{W}$. Z doesn't accumulate at Z^{la}, and as if $p| \ \mathrm{Cond}(\chi)$, we will only have a point $y \in Z^{la}$ corresponding to the automorphic representation $\pi^n(\chi)$ of section 10, we first need to enlarge a bit Z [(31)].

Proposition B.1. — *There exists $Z' \subset Z^{la}$, which accumulates at every point of Z^{la}, such that for all $z \in Z'$, we have that the Sen polynomial of ρ_z is killed by*

$$\prod_{i=1}^{3}(T - wt_i(z)).$$

Proof. — Let $z \in Z^{la}$. In particular, there exists w, ε such that $z \in \mathcal{E}_{w,\varepsilon}$ in the notations before Theorem 9.5. This $\mathcal{E}_{w,\varepsilon}$ is affinoid. Thus, by [**BC09**] Lemma 7.8.11, there exists $g : \mathcal{E}'_{w,\varepsilon} \longrightarrow \mathcal{E}_{w,\varepsilon}$ such that we have an actual representation of $G = G_{E,S}$ on a coherent torsion free sheave over $\mathcal{E}'_{w,\varepsilon}$. We can then apply [**KPX14**] Definition 6.2.11 or [**BC09**] p125 to have a Sen operator in family over $\mathcal{E}'_{w,\varepsilon}$. But Z is Zariski dense in $\mathcal{E}_{w,\varepsilon}$ thus as is its pullback Z'^{alg} in $\mathcal{E}'_{w,\varepsilon}$. Moreover, there is $Y \subset \mathcal{E}'_{w,\varepsilon}$ Zariski open and dense, on which $Z'_Y = Y \cap Z'^{alg}$ is Zariski dense, with $\rho_z = \rho_{g(z)}$ for all $z \in Y$. Thus for all $z \in Y \cap Z'^{alg}$, we have that the Sen operator is killed by

$$\prod_{i=1}^{3}(T - wt_i(z)).$$

By density, this is true for all $x \in \mathcal{E}'_{w,\varepsilon}$. Thus, for all $y \in Z'_Y = g^{-1}(Z^{la}) \cap Y$, the Sen operator of $\rho_y = \rho_{g(y)}$ is killed by the same polynomial. Using $Z' = g(Z'_Y)$ we get the result. $\square$

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(31) We could actually prove directly the following result on all Z^{la}, and even the crystabellianity of these representations, by extending results of [**BPS16**, **Bij16**] for all classical modular forms with Nebentypus, as it is done in [**PS17**]. But the following will be enough for us.

By proposition 10.21 there exists a point $y \in \mathcal{E}$, whose (semi-simplified) Galois representation is $1 \oplus \overline{\chi} \oplus \varepsilon$ and its refinement is σ (see definition 10.16). Let $A = \mathcal{O}_{\mathcal{E},y}$ be the rigid analytic local ring at y. We want to study this ring and the pseudo-character T at A. By [**BC09**] Theorem 1.4.4 and Lemma 1.8.3 for $S = A[G]/\operatorname{Ker} T$, we choose idempotents $e_\varepsilon, e_{\overline{\chi}}, e_1$ that are compatible with the involution τ given by $i \mapsto i^\perp(1)$. We thus have a generalized matrix algebra (GMA) of the form

$$\begin{pmatrix} A & A_{\varepsilon,\overline{\chi}} & A_{\varepsilon,1} \\ A_{\overline{\chi},\varepsilon} & A & A_{\overline{\chi},1} \\ A_{1,\varepsilon} & A_{1,\overline{\chi}} & A \end{pmatrix}$$

This defines $\operatorname{Ext}_T(i,j)$ and $h_{i,j} = \dim \operatorname{Ext}_T(i,j)$ for all $i \neq j \in \{1, \varepsilon, \overline{\chi}\}$. In the end, we want to study I_{tot} the total reducibility locus and this GMA.

On A/I_{tot}, we have pseudo-characters of dimension 1 (i.e. actual characters)

$$T_j : G = G_{E,S} \longrightarrow A/I_{tot}, \quad j \in \{\varepsilon, 1, \overline{\chi}\},$$

such that $T_j \otimes A/\mathfrak{m}_A = T_j \otimes k(y) = j$. From now on fix $I \supset I_{tot}$ a cofinite length ideal.

Lemma B.2. — *The Sen operator of T_j is killed by the polynomial*

$$\prod_{i=1}^{3} (T - \mathrm{wt}_i) \in A/I[T]^\Sigma.$$

Proof. — Let $y \in \mathcal{E}$. As remarked, the set Z' of B.1 accumulates at y. Fix $j \in \{1, \overline{\chi}, \varepsilon\}$ and denote $S = A[G]/\operatorname{Ker} T$. There exists M a S-module, of finite type as A-module such that $MK = K^3$ and with an exact sequence

$$0 \longrightarrow K \longrightarrow M/IM \longrightarrow T_j \longrightarrow 0,$$

such that K as a Jordan-Holder sequence with all subquotient isomorphic to T_i for $i \neq j$ (see [**BC09**] Theorem 1.5.6 and Lemma 4.3.9). Thus, it suffices to prove that M/IM as its Sen operator killed by the previous polynomial. But by [**BC09**] Lemma 4.3.7 (and because Z' accumulates at y) we can find $U \subset \mathcal{E}$ an affinoïd open containing z, in which Z' is Zariski dense, together with $\mathcal{M}$ a coherent torsion-free $\mathcal{O}_U$-module endowed with an action of G such that $\mathcal{M}(U) \otimes A \simeq M$ as $A[G]$-module, and $\mathcal{M} \otimes_{\mathcal{O}(U)} \operatorname{Frac}(\mathcal{O}(U))$ is free of rank 3, semisimple as G-module and trace $T \otimes_{\mathcal{O}(X)} \mathcal{O}(U)$. By generic semi-simplicity and generic flatness, there exists $F \subset U$ a Zariski closed subspace such that for all $x \in U \backslash F$, $\mathcal{M}_y = \mathcal{M}_y^{ss} = \rho_y$. We can change Z' by $Z' \cap (U \backslash F)$, which is still Zariski dense in U. Denote by φ the Sen operator of $D_{Sen}(\mathcal{M})$ (or $\mathcal{B} = \operatorname{End}_{\mathcal{O}(U)}(\mathcal{M}(U))$ see [**KPX14**] Definition 6.2.11 or [**BC09**] proof of lemma 4.3.3). For all $z \in Z'$, φ_z is killed by

$$P = \prod_{i=1}^{3} (T - \mathrm{wt}_i(z)),$$

by proposition B.1, and as Z' is Zariski dense, and $\mathcal{O}(U)$ is reduced we get that P kills φ on U, and reducing to A/I we get the result. $\square$

Fix the bijection between $\{1,2,3\}$ and $\{1,\overline{\chi},\varepsilon\}$ corresponding to the refinement 10.16, i.e.

$$\begin{array}{ccc} 1 & \mapsto & 1 \\ 2 & \mapsto & \overline{\chi} \\ 3 & \mapsto & \varepsilon \end{array}$$

Thus it makes sense to speak about $T_i, i \in \{1,2,3\}$.

Lemma B.3. — *For all i, the A/I-module*

$$H^0_{\varphi,\Gamma}(D_{rig}(T_i)(\delta_i^{-1}))$$

is free of rank 1.

Proof. — We will consider inductively the pseudocharacters T, $\Lambda^2 T$ and $\det T$ whose reduction is respectively $1 \oplus \overline{\chi} \oplus \varepsilon$, $\overline{\chi} \oplus \varepsilon \oplus \varepsilon\overline{\chi}$ and $\varepsilon\overline{\chi}$. In particular they are multiplicity free. Recall that for $I \supset I_{tot}$, T splits, thus also $\Lambda^2 T$, we denote $T'_i = T_1 \dots T_i$ for $i = 1,2,3$. By induction on i, it is enough to prove the result for T'_i. In particular for all i, we can find M a S-module, finite type over A, of generic rank 3 (if $i = 1,2$, rank 1 and $M = T'_3$ if $i = 3$) such that ([**BC09**] Theorem 1.5.6 and Lemma 4.3.9).

$$0 \longrightarrow K \longrightarrow M/I \longrightarrow T'_i \longrightarrow 0,$$

with K^{ss} reducing to a direct sum of $\prod_{k=1}^i T_{j_k} \neq T'_i$. As $\delta_i(p) = F_i$ and at y these values are

$$(1, \overline{\chi}(p), p^{-1}),$$

which are distincts ($|\overline{\chi}(p)| = p^{-1/2}$), the slope of $\delta_1 ... \delta_i$ is distinct from the one appearing in K. In particular

$$H^0_{\varphi,\Gamma}(D_{rig}(K(\delta_1 ... \delta_i)^{-1})) = \{0\}.$$

Thus, it suffices to show that $H^0_{\varphi,\Gamma}(D_{rig}(M(\delta_1 ... \delta_i)^{-1})))$ is free of rank 1 for every cofinite ideal J of $A = \mathcal{O}_y$. But this is assured by [**KPX14**] Theorem 6.3.9 and [**BC09**] Theorem 3.3.3 and Lemma 3.3.9. Indeed, first, by [**BC09**] Lemma 4.3.7 we can find $U \subset \mathcal{E}$ containing y an affinoid together with a coherent torsion free module $\mathcal{M}$ with an action of $G = G_{E,S}$ reducing to M on $A = \mathcal{O}_y$, which is generically free of rank 3 (or 1), and such that the trace of G on $\mathcal{M}$ coincides with $T'_i \otimes_{\mathcal{O}_\mathcal{E}} \mathcal{O}(U)$. Denote $\delta_1^{(i)} = \delta_1 \dots \delta_i$, and $H^0_{\varphi,\Gamma}(D_{rig}(-))$ is a functor as in [**BC09**] Section 3.2.2. Moreover, by [**BC09**] Lemma 3.4.2 and [**KPX14**] Theorem 6.3.9 (applied to $\mathcal{M}'^\vee$ and $\delta = \delta_1^{(i),-1}$) there exists a birational morphism (see [**BC09**] section 3.2.3)

$$\pi : U' \longrightarrow U,$$

such that the strict transform $\mathcal{M}'$ of $\mathcal{M}$ on U' is locally free, and moreover we have a map

$$D_{rig}(\mathcal{M}'^\vee) \longrightarrow R_{U'}(\delta_1^{(i),-1}) \otimes \mathcal{L},$$

whose kernel is a φ,Γ-module of rank 2 (is trivial if $i = 3$) and which is generically surjective. Moreover it is proven in the course of the proof of [**KPX14**] Theorem 6.3.9 that $H^0_{\varphi,\Gamma}(D_{rig}((\mathcal{M}'^\vee)^\vee)(\delta_1^{-1}))$ is locally free of rank 1. In particular, as these sheaves are coherent, for all $y' \in \pi^{-1}(y)$, and all cofinite length ideal J' of $\mathcal{O}_{y'}$,

$$H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}'(\delta_1^{(i),-1}) \otimes \mathcal{O}_{y'}/J')),$$

is free of rank 1. Indeed, we have the commuting diagram

$$\begin{tikzcd} H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1}))\otimes\mathcal{O}_{y'}/I' \arrow[r] \arrow[d, "f"] & H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1})\otimes\mathcal{O}_{y'}/I') \arrow[d, "red"] \\ H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1}))\otimes\mathcal{O}_{y'}/\mathfrak{m}_{y'} \arrow[r, "i"] & H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1})\otimes\mathcal{O}_{y'}/\mathfrak{m}_{y'}) \end{tikzcd}$$

where the map i is injective ([**KPX14**] eq 6.3.9.1). As the map f is non-zero, the map red is also non-zero. Thus by [**BC09**] Lemma 3.3.9,

$$H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1})\otimes\mathcal{O}_{y'}/I')$$

is free of rank one over $\mathcal{O}_{y'}/I'$. Thus by [**BC09**] Proposition 3.2.3 and Lemma 3.3.9, for all cofinite length ideal J of $\mathcal{O}_y = A$, we have that

$$H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M})(\delta_1^{(i),-1})\otimes\mathcal{O}_y/J)),$$

is free of rank 1 over A/J. □

Lemma B.4. — *Suppose that D is a φ,Γ-module of rank 1 on an artinian ring A, and with Hodge-Tate weight $k = (k_\sigma)_{\sigma\in\Sigma} \in \mathbb{Z}^\Sigma$. **Fix***

$$\delta : K^\times \longrightarrow A^\times,$$

and denote $(t_\sigma)_{\sigma\in\Sigma} \in \mathbb{Z}^\Sigma$ its Hodge-Tate weights. Suppose that

$$H^0_{\varphi,\Gamma}(D(\delta^{-1})),$$

is free of rank 1 over A. Then $D = R_A(\delta')$ with $\delta' = \delta\prod_\sigma x_\sigma^{k_\sigma - t_\sigma}$.

Proof. — Let $D = R_A(\delta')$ and by hypothesis we have a injective morphism of R_A-modules

$$R(\delta) \hookrightarrow D = R_A(\delta').$$

Let v be the image of a basis of $R(\delta)$, and denote by e a basis of D. Thus, $D' = R_A v$ is a sub-φ,Γ-module of D, isomorphic to $R_A(\delta)$. Reducing modulo $\mathfrak{m}_A$, by [**KPX14**] corollary 6.2.9 we have that $\overline{D'} = \prod_\sigma t_\sigma^{l_\sigma}\overline{D}$ for some $l_\sigma \in \mathbb{Z}$. But Γ acts on v as $\delta(\gamma)$. Moreover, using the previous equality, it also acts on $\overline{v}$ by

$$\gamma\overline{v} = \prod_\sigma LT_\sigma(\gamma)^{l_\sigma}\delta'(\gamma)\overline{v}.$$

Thus, $\overline{\delta}_{|\Gamma} = (\prod_\sigma x_\sigma\overline{\delta}')_{|\Gamma}$, which by hypothesis gives

$$l_\sigma = t_\sigma - k_\sigma.$$

Consider $M = \prod_\sigma t_\sigma^{-l_\sigma}R_A v$. Then M is saturated in D', thus $D' = M$. But as $R_A v \simeq R_A(\delta)$, $M \simeq R(\prod_\sigma x_\sigma^{-l_\sigma}\delta)$, thus, by [**KPX14**] Lemma 6.2.13,

$$\delta' = \delta\prod_\sigma x_\sigma^{k_\sigma - t_\sigma}.$$

□

Recall ([**BC09**] Lemma 8.27, that we have an injective map

$$\iota_{T,i,j} : \mathrm{Ext}_T(i,j) \hookrightarrow \mathrm{Ext}_{k[G_{E,S}]}(i,j).$$

Theorem B.5. *— Let $\rho : G \longrightarrow \mathrm{GL}_{d_i+d_j}(A/I)$ an extension of T_1 by T_i inside the image of $\iota_{T,i,1}$. Then, if p splits, for $\star = v, \overline{v}$*

$$D_{crys,\star}(\rho(\delta_{1,|\Gamma}^{-1}))^{\varphi=F_1}$$

is free of rank 1 over A/I. If p is inert,

$$D_{crys,\tau}(\rho(\delta_{1,|\Gamma}^{-1}))^{\varphi^2=F_1}$$

is free of rank 1 over A/I.

Proof. — Let's do the proof at v when p splits. Recall that 1 is the only constituent of ρ_y which has $1 = p^{\mathrm{wt}_1}F_1$ as eigenvalue for its Frobenius. By [**BC09**] Theorem 1.5.6 (2), there is an exact sequence,

$$0 \longrightarrow K \longrightarrow (M_1/IM_1 \oplus \rho_i) \longrightarrow \rho \longrightarrow 0,$$

with K^{ss} being a direct sum of $\overline{T_k}$, $k \neq 1$. Thus, $D_{crys}(K(\delta_{1|\Gamma}^{-1}))^{\varphi=F_1} = D_{crys}(T_i(\delta_{1|\Gamma}^{-1}))^{\varphi=F_1} = \{0\}$.In particular, it is enough to prove that

$$D_{crys}(M_1(\delta_{1|\Gamma}^{-1}))^{\varphi=F_1}$$

is free of rank 1 over A. We will use the same devissage as in B.3. By [**BC09**] Lemma 4.3.9, there exists $M = M_1 \oplus N_1$ such that $MK = K^3$ a sub-$A[G]$ module of K^3 of finite type over A. Extending this module to an affinoid $U \subset \mathcal{E}$ containing y, and using the accumulation of Z' at y (Proposition B.1), we can find a birational morphism $\pi : U' \longrightarrow U$ and $\mathcal{M}'$ the strict transform of $\mathcal{M}$, locally free on U', for which the conclusion of [**KPX14**] Theorem 6.3.9 for $(\mathcal{M}')^\vee$ and δ_1^{-1} applies. In particular

$$H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1}))$$

is locally free of rank one on U'.

As in Lemma B.3 we can specialize at $\mathcal{O}_{y'}$ for every y' above $y \in U$. But we have the commuting diagram

$$\begin{array}{ccc} H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1})) \otimes \mathcal{O}_{y'}/I' & \longrightarrow & H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1}) \otimes \mathcal{O}_{y'}/I') \\ \Big\downarrow f & & \Big\downarrow red \\ H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1})) \otimes \mathcal{O}_{y'}/\mathfrak{m}_{y'} & \xrightarrow{\ i\ } & H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1}) \otimes \mathcal{O}_{y'}/\mathfrak{m}_{y'}) \end{array}$$

where the map i is injective ([**KPX14**] eq 6.3.9.1), the map f is non-zero, thus the map red is also non-zero. By [**BC09**] Lemma 3.3.9,

$$H^0_{\varphi,\Gamma}(D_{rig}(\mathcal{M}')(\delta_1^{-1}) \otimes \mathcal{O}_{y'}/I')$$

is free of rank one over $\mathcal{O}_{y'}/I'$ for all $y' \in \pi^{-1}(y)$ and I' of cofinite length. Thus the hypothesis of [**BC09**] Proposition 3.2.3 are satisfied, and by [**BC09**] Lemma 3.3.9 again,

$$H^0_{\varphi,\Gamma}(D_{rig}(M \otimes A/I)(\delta_1^{-1}))$$

is free of rank 1. In particular we have an injection of $R_{A/I}$-modules,

$$0 \longrightarrow R_A \longrightarrow D_{rig}(M \otimes A/I)(\delta_1^{-1}) \longrightarrow Q \longrightarrow 0.$$

Moreover, as the reduction to $A/\mathfrak{m}_A$ of $D_{crys}(M(\delta_1^{-1}))^{\varphi=1}$ is of rank 1, using the functor D_{cris}, we have that $D_{cris}(1) \subset D_{cris}(M(\delta_1^{-1}))^{\varphi=1}$ and thus $D_{crys}(Q)^{\varphi=1} = \{0\}$. In particular $D_{crys}(M(\delta^{-1}))^{\varphi=1} = D_{crys}(1)$ is free of rank 1 over A, and thus

$$D_{crys}(M(\delta_{1|\Gamma})^{-1})^{\varphi=F_1}$$

is free of rank 1 over A. The same proof remains valid in the case where p is inert, as $1 = p^{wt_1}F_1$ is also the first and only constituent of ρ_y, and by duality in the inert case, as $\rho_{y,\overline{v}} = \rho_{y,v}^\vee \varepsilon^{-1}$, whose refinement at y is given by $\varepsilon(p)^{-1}(1, \chi_v(p), \varepsilon(p))^\vee = (1, \chi_{\overline{v}}(p), \varepsilon(p))$, thus starts by 1, thus the same proof as for v also applies for $\overline{v}$. □

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