Théorie des Nombres TD8

M2 AAG 2021-2022

Exercice 1 (Ray class fields). Let F be a number field.

1. Show that for varying moduli m,

 $\mathfrak{m} \mapsto F(\mathfrak{m}),$

is increasing. Is it strictly increasing ? Is it strictly increasing if we impose all the real places in \mathfrak{m} (i.e. only look at *narrow/extended* moduli) ?

2. Let $F = \mathbb{Q}(\sqrt{3})$. Show that 11 is split and denote \mathfrak{p}_{11} a prime above 11, and denote $\mathfrak{m} = \mathfrak{p}_{11}(\infty) = F_{\infty} \times \mathfrak{p}_{11}$. Show that the ray class field $F(\mathfrak{m})$ is equal to the extended Hilbert class field $F(\infty)$.

Exercice 2 (Local and global conductors). Let F be a number field, and L/F a finite abelian extension.

Definition 1. The *(global) conductor* $\mathfrak{f}(L/F)$ of L/F is the smallest modulus¹ $\mathfrak{m} = F_{\infty} \prod_{v \nmid \infty} \mathfrak{m}_{v}^{n_{v}}$ for F such that $L \subset F(\mathfrak{m})$ (equivalently, such that $F^{\times}V_{\mathfrak{m}} \subset F^{\times}N_{L/F}(\mathbb{A}_{L}^{\times})$, equivalently $V_{\mathfrak{m}} \subset F^{\times}N_{L/F}(\mathbb{A}_{L}^{\times})$).

1. Is the conductor of $F(\mathfrak{m})$ necessarily \mathfrak{m} ?

Recall that if M/K is an extension of local (*p*-adic) fields, the *local conductor* $n_{M/K}$ for M/K is the smallest n such that $U_K^n \subset N_{M/K}(M^{\times})$. We denote $\mathfrak{f}(M/K) = \mathfrak{m}_K^{n_{M/K}}$.

2. Show that if M/K is a local extension, then M/K is unramified iff $n_{M/K} = 0$ iff $\mathfrak{f}(M/K) = \mathcal{O}_M$.

¹To simplify we assume that our moduli contain all (real) places at infinity, i.e. we allow ramification at infinity. In this situation we should probably speak about finite conductor, or Extended/Narrow conductor, but for simplicity we ignore this issue.

If $\alpha = (\alpha_v)_{v \in \Sigma_F} \in \mathbb{A}_F^{\times}$ we denote $[\alpha_v] \in \mathbb{A}_F^{\times}$ the idele concentrated at the place v, i.e.

$$[\alpha_v] = (1, 1, \dots, 1, \underbrace{\alpha_v}_{\text{place } v}, 1, \dots, 1, \dots).$$

We denote by $N = N_{L/F}(\mathbb{A}_L^{\times})$ and by $\overline{\cdot} : \mathbb{A}_F^{\times} \longrightarrow \mathbb{A}_F^{\times}/F^{\times}$.

- 3. Show that for all place of F, $\overline{[\alpha_v]} \in \overline{N}$ (i.e. $[\alpha_v] \in F^{\times}N$) iff $\alpha_v \in N_{L_v/F_v}(L_v^{\times})$.
- 4. Deduce that if for all $v, [\alpha_v] \in F^{\times}N$, then $\alpha \in N$.

If K is a local field, and M is a (finite) product of local fields M_w over K (e.g. $K = F_v$, and $M = L_v := L \otimes_K K_v = \prod_{w|n} L_w$) we define $n_{M/K} = \min_w n_{M_w/K}$ and, thus, $\mathfrak{f}(M/K) = \mathfrak{m}_K^{n_{M/K}} = \operatorname{pgcd}_w \mathfrak{f}(M_w/K)$.

5. Show that the global conductor is the product of the local ones, i.e.

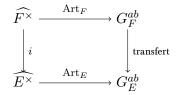
$$\mathfrak{f}(L/F) = F_{\infty}^{\times} \prod_{v \nmid \infty} \mathfrak{f}(L_v/F_v).$$

6. Extend this by taking care of places at infinity.

Exercice 3 (Transfert). Let K/E/F be a tower of finite extensions, with K/E and K/F finite abelian.

1. Is K/F necessarily abelian if Galois ?

The infinite reformulation of local class field theory says that $\widehat{F^{\times}} \simeq G_F^{ab}$ and $\widehat{E^{\times}} \simeq G_E^{ab}$. Moreover there is a clear continuous map $F^{\times} \subset E^{\times}$ which induces $\widehat{F^{\times}} \subset \widehat{E^{\times}}$. Thus there is a transfert map which makes the following diagram commute



The goal is to describe this map.

2. Show that there is a natural map $F^{\times}/N_{K/F}(K^{\times}) \longrightarrow E^{\times}/N_{K/E}(E^{\times})$.

3. Let G be a group, and H be a subgroup of finite index, and let X = G/H. For all $x \in X$, choose a representative $\tilde{x} \in G$ of x. Show that

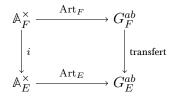
$$Ver: \begin{array}{ccc} G & \longrightarrow & H^{ab} \\ s & \longmapsto & \prod_{x \in X} h_x(s) \end{array}$$

where $h_x(s) \in H$ satisfies $s\tilde{x} = \tilde{x'}h_x(s)$ with $x' \in X$ is a well defined group morphism which doesn't depend on representatives of X.

- 4. Deduce that Ver induces a group morphism $Ver: G^{ab} \longrightarrow H^{ab}$.
- 5. Show that if we have a Galois extension M of K, and we denote $G = \operatorname{Gal}(M/K), H = \operatorname{Gal}(M/L)$ and choose representative for $g \in G$ of $< g > \backslash G/H \{\sigma_1, \ldots, \sigma_s\}$ and we denote by, for all i, f_i the minimal power such that $\sigma_i g^{f_i} \sigma_i^{-1} \in H$, then

$$V(g) = \prod_{i=1}^r \sigma_i^{-1} g^{f_i} \sigma_i.$$

6. Deduce that if E/F is an extension of number fields,



is commutative.

- 7. Deduce that for any local extension, the local diagram is commutative.
- 8. We assume the purely group-theoretic result for now :

Proposition 1. If G is a finite group and H = [G : G] is the commutator subgroup, then Ver = 0.

Show that every ideal in F becomes principal in E = F(1), the Hilbert class field of F.

Exercice 4 (Golod-Shafarevich). Show that the 2-Class field tower of $\mathbb{Q}(i\sqrt{2 \times 3 \times 5 \times 7 \times 11 \times 13})$ is infinite (cf. Georges Gras p188).