Théorie des Nombres TD8

M2 AAG 2021-2022

Exercice 1 (Ray class fields). Let F be a number field.

1. Show that for varying moduli $\mathfrak{m}$,

$$\mathfrak{m} \mapsto F(\mathfrak{m}),$$

   is increasing. Is it strictly increasing ? Is it strictly increasing if we impose all the real places in $\mathfrak{m}$ (i.e. only look at *narrow/extended* moduli) ?

2. Let $F = \mathbb{Q}(\sqrt{3})$. Show that 11 is split and denote $\mathfrak{p}_{11}$ a prime above 11, and denote $\mathfrak{m} = \mathfrak{p}_{11}(\infty) = F_\infty \times \mathfrak{p}_{11}$. Show that the ray class field $F(\mathfrak{m})$ is equal to the extended Hilbert class field $F(\infty)$.

Exercice 2 (Local and global conductors). Let F be a number field, and L/F a finite abelian extension.

Definition 1. The *(global) conductor* $\mathfrak{f}(L/F)$of L/F is the smallest modulus[1] $\mathfrak{m} = F_\infty \prod_{v \nmid \infty} \mathfrak{m}_v^{n_v}$ for F such that $L \subset F(\mathfrak{m})$ (equivalently, such that $F^\times V_\mathfrak{m} \subset F^\times N_{L/F}(\mathbb{A}_L^\times)$, equivalently $V_\mathfrak{m} \subset F^\times N_{L/F}(\mathbb{A}_L^\times)$).

1. Is the conductor of $F(\mathfrak{m})$ necessarily $\mathfrak{m}$?

Recall that if M/K is an extension of local (p-adic) fields, the *local conductor* $n_{M/K}$ for M/K is the smallest n such that $U_K^n \subset N_{M/K}(M^\times)$. We denote $\mathfrak{f}(M/K) = \mathfrak{m}_K^{n_{M/K}}$.

2. Show that if M/K is a local extension, then M/K is unramified iff $n_{M/K} = 0$ iff $\mathfrak{f}(M/K) = \mathcal{O}_M$.

[1] To simplify we assume that our moduli contain all (real) places at infinity, i.e. we allow ramification at infinity. In this situation we should probably speak about finite conductor, or Extended/Narrow conductor, but for simplicity we ignore this issue.

If $\alpha = (\alpha_v)_{v \in \Sigma_F} \in \mathbb{A}_F^\times$ we denote $[\alpha_v] \in \mathbb{A}_F^\times$ the idele concentrated at the place v, i.e.

$$[\alpha_v] = (1, 1, \dots, 1, \underbrace{\alpha_v}_{\text{place } v}, 1, \dots, 1, \dots).$$

We denote by $N = N_{L/F}(\mathbb{A}_L^\times)$ and by $\bar{\cdot} : \mathbb{A}_F^\times \longrightarrow \mathbb{A}_F^\times / F^\times$.

3. Show that for all place of F, $\overline{[\alpha_v]} \in \overline{N}$ (i.e. $[\alpha_v] \in F^\times N$) iff $\alpha_v \in N_{L_v/F_v}(L_v^\times)$.

4. Deduce that if for all v, $[\alpha_v] \in F^\times N$, then $\alpha \in N$.

If K is a local field, and M is a (finite) product of local fields M_w over K (e.g. $K = F_v$, and $M = L_v := L \otimes_K K_v = \prod_{w|n} L_w$) we define $n_{M/K} = \min_w n_{M_w/K}$ and, thus, $\mathfrak{f}(M/K) = \mathfrak{m}_K^{n_{M/K}} = \mathrm{pgcd}_w\, \mathfrak{f}(M_w/K)$.

5. Show that the global conductor is the product of the local ones, i.e.

$$\mathfrak{f}(L/F) = F_\infty^\times \prod_{v \nmid \infty} \mathfrak{f}(L_v/F_v).$$

6. Extend this by taking care of places at infinity.

Exercice 3 (Transfert). Let $K/E/F$ be a tower of finite extensions, with K/E and K/F finite abelian.

1. Is K/F necessarily abelian if Galois ?

The infinite reformulation of local class field theory says that $\widehat{F^\times} \simeq G_F^{ab}$ and $\widehat{E^\times} \simeq G_E^{ab}$. Moreover there is a clear continuous map $F^\times \subset E^\times$ which induces $\widehat{F^\times} \subset \widehat{E^\times}$. Thus there is a transfert map which makes the following diagram commute

$$\begin{array}{ccc} \widehat{F^\times} & \xrightarrow{\mathrm{Art}_F} & G_F^{ab} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \text{transfert}} \\ \widehat{E^\times} & \xrightarrow{\mathrm{Art}_E} & G_E^{ab} \end{array}$$

The goal is to describe this map.

2. Show that there is a natural map $F^\times / N_{K/F}(K^\times) \longrightarrow E^\times / N_{K/E}(E^\times)$.

3. Let G be a group, and H be a subgroup of finite index, and let $X = G/H$. For all $x \in X$, choose a representative $\tilde{x} \in G$ of x. Show that

$$Ver : \begin{array}{ccc} G & \longrightarrow & H^{ab} \\ s & \longmapsto & \prod_{x \in X} h_x(s) \end{array}$$

where $h_x(s) \in H$ satisfies $s\tilde{x} = \tilde{x'} h_x(s)$ with $x' \in X$ is a well defined group morphism which doesn't depend on representatives of X.

4. Deduce that Ver induces a group morphism $Ver : G^{ab} \longrightarrow H^{ab}$.

5. Show that if we have a Galois extension M of K, and we denote $G = \mathrm{Gal}(M/K), H = \mathrm{Gal}(M/L)$ and choose representative for $g \in G$ of $< g > \backslash G/H$ $\{\sigma_1, \ldots, \sigma_s\}$ and we denote by, for all i, f_i the minimal power such that $\sigma_i g^{f_i} \sigma_i^{-1} \in H$, then

$$V(g) = \prod_{i=1}^{r} \sigma_i^{-1} g^{f_i} \sigma_i.$$

6. Deduce that if E/F is an extension of number fields,

$$\begin{array}{ccc} \mathbb{A}_F^\times & \xrightarrow{\mathrm{Art}_F} & G_F^{ab} \\ \Big\downarrow i & & \Big\downarrow \text{transfert} \\ \mathbb{A}_E^\times & \xrightarrow{\mathrm{Art}_E} & G_E^{ab} \end{array}$$

is commutative.

7. Deduce that for any local extension, the local diagram is commutative.

8. We assume the purely group-theoretic result for now :

Proposition 1. *If G is a finite group and $H = [G : G]$ is the commutator subgroup, then $Ver = 0$.*

Show that every ideal in F becomes principal in $E = F(1)$, the Hilbert class field of F.

Exercice 4 (Golod-Shafarevich). Show that the 2-Class field tower of $\mathbb{Q}(i\sqrt{2 \times 3 \times 5 \times 7 \times 11 \times 13})$ is infinite (cf. Georges Gras p188).