Théorie des Nombres TD5

M2 AAG 2021-2022

Exercise 1 (Gauss' sums). Let F a finite extension of $\mathbb{Q}_p$, $\omega : \mathcal{O}_F^\times \longrightarrow S^1$ and $\psi : \mathcal{O}_F \longrightarrow \mathbb{C}^\times$ be (resp.) a multiplicative and additive character, and we fix dx and $d^\times x$ Haar mesures on F and $F^\times$. We define the Gauss' sum to be

$$G(\omega, \psi) := \int_{\mathcal{O}_F^\times} \omega(u)\psi(u) d^\times u.$$

Denote by n and d the respective conductors[1] of ω and ψ. Let $c \in F^\times$ such that $c|x|^{-1}dx = d^\times x$.

1. Show that this is actually a finite sum, which coincides with an actual Gauss sum up to a constant.
2. Show that if $d < n$ then show $G(\omega, \psi) = 0$.
3. Show that if $d = n$ then $|G(\omega, \psi)|^2 = c \operatorname{vol}(\mathcal{O}_F, dx) \operatorname{vol}(U_r, d^\times x)$.

Exercise 2 (Epsilon factors for ramified characters). Let $F/\mathbb{Q}_p$ be a finite extension with uniformizer π. Let $\chi : F^\times \longrightarrow \mathbb{C}^\times$ be a continuous character of conductor n.

1. Show that χ is of the form $(\chi^u \circ u_\pi)|.|^s$ for $s \in \mathbb{C}$ and $\chi^u : \mathcal{O}_F^\times \longrightarrow S^1$ with $u_\pi : F^\times \longrightarrow \mathcal{O}_F^\times$ explicit depending on π. Is s well defined ?
2. Show that we can write $\chi = \chi_0|.|^s$ with $\chi_0 : F^\times \longrightarrow S^1$. Is such a decomposition unique ?

Now we let $\psi : F \longrightarrow S^1$ be a non trivial additive character ψ of conductor d. Our goal is to calculate the epsilon factor $\varepsilon(\chi, \psi, dx) = \varepsilon(\chi, \psi)$. We choose dx the auto-dual Haar mesure on F.

3. Let $g = \psi \mathbb{1}_{\pi^{d-n}\mathcal{O}_F}$. Calculate the Fourier transform $\hat{g}$ of g.
4. What are $L(\chi)$ and $L(\chi^\vee)$, with $\chi^\vee = \chi^{-1}|.|$?
5. Assume $n = 0$, calculate $\varepsilon(\psi, \chi)$.
6. Assume $n > 0$, calculate $\varepsilon(\psi, \chi)$.

From now on we assume that $\psi = \bigotimes_v \psi_v : \mathbb{A}_F \longrightarrow \mathbb{C}^\times$ satisfies $\psi(F) = 1$ and $\psi_v(\mathcal{O}_v) = 1$ for almost all v, and $\chi = \bigotimes_v \chi_v : \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$ such that $\chi(F^\times) = 1$ and χ is unramified almost everywhere.

7. Show that

$$\varepsilon(\psi, \chi) := \prod_v \varepsilon_v(\psi_v, \chi_v) \in \mathbb{C}^\times,$$

is well defined.

Exercise 3 (Real epsilon factors). Here we assume $F = \mathbb{R}$ and set $\psi : t \in \mathbb{R} \mapsto e^{-2\pi i t}$, and we take the autodual Haar mesure for ψ.

1. Let f such that $f(x) = e^{-\pi x^2}$. Calculate $\hat{f}$.
2. Calculate $Z(f, |.|^s)$. Deduce $\varepsilon(|.|^s, \psi)$.
3. Let g such that $g(x) = xe^{-\pi x^2}$. Calculate $\hat{g}$.

[1] We say that an additive (resp. multiplicative) character ψ (ω) has conductor r (n) if its conductor is (π^r) (resp. $1 + (\pi^n)$) for π a uniformizer

4. Calculate $Z(g, \operatorname{sgn}|.|^s)$. Deduce that $\varepsilon(\psi, \operatorname{sgn}|.|^s) = -i$.

5. Deduce that $\frac{Z(g,\omega)}{L(\omega)}$ is holomorphic everywhere, i.e. for all continuous character $\omega : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$.

Exercise 4 (Complex epsilon factors). Here we assume $F = \mathbb{C}$ and we set $\psi : z \in \mathbb{C} \mapsto e^{-2i\pi(z+\overline{z})}$. We take the autodual Haar mesure for ψ.

1. Show that all continuous characters $\omega : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ are of the shape $\omega = |.|^s\theta_n$, for $s \in \mathbb{C}$, $n \in \mathbb{Z}$, with
$$\theta_n : re^{i\theta} \mapsto e^{in\theta}.$$

2. Let
$$f_n(z) = \begin{cases} \overline{z}^n e^{-2\pi z\overline{z}} & \text{if } n \geqslant 0 \\ z^{-n}e^{-2\pi z\overline{z}} & \text{if } n < 0 \end{cases}$$
Show that $\hat{f_n} = i^{|n|}f_{-n}$. ***Hint : first calculate*** $\hat{f}_0$ ***and then apply the operator*** $d\overline{z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$.

3. Calculate $Z(f_n, |.|^s\theta_n)$.

4. Deduce that the epsilon factor $\varepsilon(\psi, \theta_n|.|^s) = i^{-|n|}$.

Exercise 5 (Non archimedian tempered distributions). Let F be a finite extension of $\mathbb{Q}_p$. Denote $S(F)$ the Schwartz functions on F (i.e. locally constant functions with compact support). Denote $S(F)'$ its continuous dual, called ***tempered*** distributions. Let $w : F^\times \longrightarrow \mathbb{C}^\times$. We denote $w = w_0|.|^s$ with w_0 unitary. We have an action of $F^\times$ on $S(F)$ by $a \cdot f(z) = f(az)$, and on $S(F)'$ by
$$< a \cdot z, f > := < z, a^{-1} \cdot f > .$$

1. Show that for $\Re s > 0$, the distribution $z(\omega_0|.|^s)$ defined by
$$< z(\omega_0|.|^s), f >= \int_{F^\times} f(x)\omega(x)d^\times x,$$
is in $S(F)'$, and actually in $S(F)'(\omega) := \{z \in S(F)' |\ a \cdot z = \omega(a)z\}$.

Let $C_c^\infty(F^\times)$ the subspace of $S(F)$ of functions with compact support in $F^\times$, and denote $C_c^\infty(F^\times)'$ its dual.

2. Show that we have an exact sequence
$$0 \longrightarrow S(F)'_0 \longrightarrow S(F)' \xrightarrow{res} C_c^\infty(F^\times)' \longrightarrow 0,$$
with $S(F)'_0 := \ker(res)$ the distribution that are ***supported at*** 0. Show moreover that the following sequence is exact
$$0 \longrightarrow S(F)'_0(\omega) \longrightarrow S(F)'(\omega) \longrightarrow C_c^\infty(F^\times)'(\omega).$$

3. Show that for all ω, $C_c^\infty(F^\times)'(\omega)$ is of dimension 1, spanned by $z(\omega) := \int_{F^\times} \cdot\omega(x)d^\times x$.

4. Give an invariant distribution $\delta_0 \in S(F)'_0$ and show that $S(F)'_0(1) = \mathbb{C}\delta_0$, and for all $\omega \neq 1$, $S(F)'_0(\omega) = \{0\}$.

5. For $f \in S(F)$ let $\nabla(f) : x \mapsto f(x) - f(\pi^{-1}x)$. Show that
$$< z_0(\omega), f > := \int_{F^\times} \nabla(f)(x)\omega(x)d^\times x$$
defines a distribution in $S(F)'(\omega)$.

6. Show that for all ω, $\dim S(F)'(\omega) = 1$. What is the relation between $z_0(\omega)$ and $z(\omega)$ if $s = \Re(w) > 0$?