On the geometry of the Pappas-Rapoport models in the (AR) case

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1 Introduction

In the last 50 years at least, Shimura varieties have played a central role in the Langlands program. Most of the time, these varieties can be thought of as moduli spaces of abelian varieties over $\mathrm{Spec}(\mathbb{Q})$. It turns out that for arithmetic applications it is sometimes desirable to have an integral structure on these spaces to be able to use their reduction modulo primes p. Such integral structures have been studied first by Deligne-Rapoport ([DR73]) and Katz-Mazur ([KM85]) for the modular curve and have been largely studied since then.

Obviously, there are a priori a lot of possible choices for these integral models, but there is a natural way to choose one by extending the moduli problem over $\mathrm{Spec}(\mathbb{Z})$ (or some localisation of it). This *natural* strategy has been extensively studied and most of the results in the P.E.L. case can be found in works of Lan ([Lan13]) as long as the moduli problem is *unramified*. An extra difficulty appears when we allow ramification in the moduli problem. In this case it has been realized a long time ago in works of Pappas and Rapoport (see e.g. [PR03]) that the natural moduli problem has bad geometric properties. In their work, Pappas and Rapoport suggested to study a slightly different integral model than the *natural* one, by adding to the moduli problem (parametrizing abelian schemes) an extra linear data of a flag of the Hodge filtration (with some restricting properties). This model is refered to as the *splitting model*, or (as we call it) Pappas-Rapoport model. In some sense, this model should be thought of as a blow-up of the natural model along some of its singularities. The goal of this article is to study the geometry of this model, and in particular of its special fiber.

Let us be more precise. As in [BH22], we focus on the P.E.L cases of type A and C, allowing some ramification. More precisely, we consider quasi-split unitary or symplectic groups over a ramified number field. In case C, we studied all cases in [BH22], proving that the model is smooth, and the ordinary locus is dense in the special fiber. In case A, we have a CM field F over a totally real field F_0, and we studied in [BH22] the geometry of the model and its special fiber at p under the assumption that F/F_0 was unramified at p. In this case we showed also that the model is smooth and the μ-ordinary locus is dense in the special fiber. Both of this results were expected since [PR05], and proved in some cases. It was clear, since the PhD thesis of Kramer ([Kra03]) that the model could not be smooth if we allow F/F_0 to be ramified. In this article, we study the geometry of the special fiber of the Pappas-Rapoport model under this assumption (which is referred to as the (AR) case). We prove that the special fiber is stratified by an explicit poset with a combinatorial description, and in particular we have a description of the irreducible components of the special fiber. Moreover we prove the closure relations for this stratification. Note that the Rapoport locus coincide with one (maximal) stratum of the special fiber. Let us give a precise formulation when $F_0 = \mathbb{Q}$, i.e. when F is quadratic imaginary, and p is a ramified prime, π a uniformizer of $F_p := F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Let a, b be integers with $a \leq b$, and let Y be the Pappas-Rapoport model over O_{F_p} (see Definition 2.1) for the unitary group $GU(a,b)_{F/\mathbb{Q}}$, and let X be its special fiber. The entire point of the Pappas-Rapoport model in this case is that over Y, and thus over X, there is a locally direct factor $\omega_1 \subseteq \omega$ of rank a, where ω is the conormal sheaf of the universal abelian scheme. There is also a second locally direct factor $\omega_2 \subseteq \omega$ of rank b, which is obtained from ω_1 and the polarization (see Definition 2.3). Moreover the universal abelian scheme has an action of O_F, thus so does ω.

For every $0 \leq h \leq \ell \leq a$, set

$$X_{h,\ell} := \{x \in X \mid \dim \pi\omega = h, \dim \omega_1 \cap \omega_2 = \ell\}.$$

This is a locally closed subscheme of X. For example $X_{a,a}$ is the (generalised) Rapoport locus. Our first result is the following,

Theorem 1.1. *Assume $p \neq 2$. For all $h \leq \ell$, the stratum $X_{h,\ell}$ is non empty, smooth, and equidimensional of dimension $ab - \frac{(\ell-h)(\ell-h+1)}{2}$. Moreover we have the closure relations,*

$$\overline{X_{h,\ell}} = \coprod_{0 \leq h' \leq h \leq \ell \leq \ell' \leq a} X_{h',\ell'}.$$

In particular X is not smooth, and the smooth locus is the union of the $X_{h,h}$ for $0 \leq h \leq a$. Moreover Y is flat over O_F, normal, and X is a local complete intersection.

When $F_0 \neq \mathbb{Q}$, one has a similar description (the index of the stratification being more complicated), all computations reducing to the previous case. Let us stress that in the case of an unramified prime there is only one open stratum (the Rapoport locus) as in the cases considered in [BH22]. In particular, in the situation considered here (i.e. the (AR) case), there is no chance that the Rapoport locus or the μ-ordinary locus is Zariski dense. If $p = 2$, we have partial similar results, which depend on the class of some pairing. In particular it can happen that some of the strata are empty (see Proposition 6.5), and in general the smooth locus is more complicated than the union of the open strata (see Section 6).

We can then study how the stratification studied previously interacts with "classical" stratification, for example the one induced by the *partial* Hasse invariants. The general result is likely to be overly complicated for combinatorial reasons, so let us describe the situation when $F_0 = \mathbb{Q}, p \neq 2$ and $(a,b) = (1,n)$, $n \geq 1$. In this case the previous stratification gives only 3 strata, $R = X_{1,1}$, the Rapoport locus, $B = X_{0,0}$ the other open strata and the intersection of their closures $P = X_{0,1}$. We have two partal Hasse invariants $\mathrm{hasse}_1, \mathrm{hasse}_2$ (see Definition 4.1) and we stratify further these 3 strata depending on the vanishing of these invariants. It turns out that there are restrictions on these vanishings, and we end up with 6 strata, refining the previous stratification :

$$R = R_0 \sqcup R_1 \sqcup R_2, \quad B = B_0 \sqcup B_1 \sqcup B_2, \quad P = P_0 \sqcup P_1 \sqcup P_2,$$

with $R_0 = X^{ord}$ the μ-ordinary locus, R_0, P_0, B_0 the locus of non vanishing of both the partial Hasse invariants, and R_2, P_2 the locus of vanishing of both partial Hasse invariants (see Section 4). We then have the following description, maybe surprising for $\overline{B_1}$ and when $n = 1$.

Theorem 1.2. *If $n = 1, 2$ the strata R_1, P_1 are empty. If $n = 1$ we have,*

$$\overline{X^{ord}} = X^{ord} \cup P_0, \quad \overline{R_2} = R_2 \cup P_2, \quad \overline{B_0} = B_0 \cup B_1 \cup B_2 \cup P_0 \cup P_2,$$

while X^{ord}, R_2, B_0 are open, and P_0, P_2, B_1, B_2 are closed.
If $n \geq 2$, then B_2 and P_2 are closed, and we have the closure relations

$$\overline{X^{ord}} = X^{ord} \cup_{i=1}^2 R_i \cup_{i=0}^2 P_i \qquad \overline{R_2} = R_2 \cup P_2$$

$$\overline{B_0} = \cup_{i=0}^2 B_i \cup_{i=0}^2 P_i \qquad \overline{B_1} = B_1 \cup \; P_1 \cup P_2 \qquad \overline{P_0} = \cup_{i=0}^2 P_i.$$

If $n \geq 3$, one has moreover

$$\overline{R_1} = \cup_{i=1}^2 R_i \cup_{i=1}^2 P_i \qquad \overline{P_1} = P_1 \cup P_2.$$

We expect that this combinatorial description of the special fiber will relate to more classical geometric varieties, and hopefully that we will be able to prove some cohomological vanishing of modular forms using the geometry of the model. As a first step, we can already prove that the extra irreducible components in the special fiber, i.e. those which are disjoint from the (generalised) Rapoport locus, do not contribute to modulo p modular forms in sufficiently regular weights. Namely assume $F_0 = \mathbb{Q}$ and let $\kappa = (k_1 \geq \cdots \geq k_a, \ell_1 \geq \cdots \geq \ell_b) \in \mathbb{Z}^{a+b}$ be a weight (see section 3). Then we have the following result.

Theorem 1.3. *If $h < a$ and if we cannot find $\{i_1 < \cdots < i_{a-h}\} \subset \{1, \ldots, a\}$ such that*

$$k_{i_1} = \cdots = k_{i_{a-h}} \leq \ell_{b-h+1},$$

then $H^0(\overline{X_{h,h}}, \omega^\kappa) = 0$.

We hope to generalise this result to higher cohomology and less restrictive weights. In [SYZ21], similar results on the geometry of the mod p fibers of Shimura varieties are proven for the Pappas-Zhu model and its EKOR stratification. It would be interesting to know if our results are related to theirs.

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2 Case of a quadratic imaginary F

2.1 Definition of the variety

Let F be a complex quadratic extension of $\mathbb{Q}$, and assume that p is ramified in F. Let F_p be the completion of F at p, and π a uniformizer of F_p. We write σ_1, σ_2 the embeddings of F_p into $\overline{\mathbb{Q}_p}$, and let us define $\pi_i = \sigma_i(\pi)$. Let a, b be integers with $a \leq b$, and define $m = a + b$.

Definition 2.1. Let Y be the moduli space over O_{F_p} whose R-points are couples $(A, \lambda, \iota, \eta, \omega_1)$, where

- A is an abelian scheme over R of dimension m
- λ is a polarization
- $\iota : O_F \to End(A)$, making the Rosati involution and the complex conjugation compatible

- η is a level structure
- $\omega_1 \subseteq \omega_A$ is a locally direct factor of rank a, stable by O_F
- O_F acts by σ_1 on ω_1, and by σ_2 on ω_A/ω_1.

Let $\mathcal{E} = H^1_{dR}(A)$; it is a locally free sheaf on Y of rank $2m$. If has an action of O_F, and is locally free of rank m over $O_Y \otimes_{\mathbb{Z}} O_F$. The Hodge filtration is $\omega_A \subseteq \mathcal{E}$. The sheaf $\mathcal{E}$ has an action of O_{F_p}, and let $[a]$ be the action of a on $\mathcal{E}$ for every $a \in O_{F_p}$. The last condition implies that $([\pi]-\pi_1)\omega_1 = 0$ and $([\pi]-\pi_2)\omega \subseteq \omega_1$.

Thanks to the polarization, one has a perfect pairing on $<,>$ on $\mathcal{E}$. The condition between the Rosati involution and the complex conjugation implies that for all $x, y \in \mathcal{E}$ one has

$$< [a] \cdot x, y >=< x, [\overline{a}] \cdot y >$$

The Hodge filtration is totally isotropic for this pairing. Moreover, the above relation implies that

$$\mathcal{E}[[\pi]-\pi_i]^{\perp} = \mathcal{E}[[\pi]-\pi_i]$$

where $\mathcal{E}[[\pi]-\pi_i]$ consists of the elements of $\mathcal{E}$ killed by $[\pi]-\pi_i$.

Remark 2.2. Since $\mathcal{E}[[\pi]-\pi_1]^{\perp} = \mathcal{E}[[\pi]-\pi_1]$, one has a perfect pairing between $\mathcal{E}[[\pi]-\pi_1]$ and $\mathcal{E}/\mathcal{E}[[\pi]-\pi_1]$. This last sheaf is isomorphic to $\mathcal{E}[[\pi]-\pi_2]$ via the multiplication by $[\pi]-\pi_1$. One has thus an induced pairing between $\mathcal{E}[[\pi]-\pi_1]$ and $\mathcal{E}[[\pi]-\pi_2]$, given by the formula

$$\{([\pi]-\pi_2)x, ([\pi]-\pi_1)y\} :=< ([\pi]-\pi_2)x, y >$$

Definition 2.3. Let us define $\omega_2 \subseteq \mathcal{E}$ by the formula

$$\omega_2 = (([\pi]-\pi_2)^{-1}\omega_1)^{\perp}$$

Proposition 2.4. *The sheaf ω_2 is locally free of rank b, and one has $\omega_2 \subseteq \omega$. Moreover, one has*

$$([\pi]-\pi_2) \cdot \omega_2 = 0 \qquad ([\pi]-\pi_1) \cdot \omega \subseteq \omega_2$$

Proof. From the properties satisfied by ω_1, one has $\omega \subseteq ([\pi]-\pi_2)^{-1}\omega_1$. Taking the orthogonal of this relation (ans using that $\omega^{\perp} = \omega$), one finds the relation $\omega_2 \subseteq \omega$.
One has $\mathcal{E}[[\pi]-\pi_2] \subseteq \omega_2^{\perp}$, and taking the orthogonal gives $\omega_2 \subseteq \mathcal{E}[[\pi]-\pi_2]$. In other words, $([\pi]-\pi_2) \cdot \omega_2 = 0$.
For the last point, we first claim that $([\pi]-\pi_1) \cdot \omega_1^{\perp} = \omega_2$. Indeed, let $x \in \omega_2$; since it belongs to $\mathcal{E}[[\pi]-\pi_2]$ there exists $x' \in \mathcal{E}$ such that $x = ([\pi]-\pi_1)x'$. Then

$$\begin{aligned} x \in \omega_2 \Leftrightarrow &\quad < x, y >= 0 \quad \forall y \in ([\pi]-\pi_2)^{-1}\omega_1 \\ \Leftrightarrow &\quad < x', ([\pi]-\pi_2)y >= 0 \quad \forall y \in ([\pi]-\pi_2)^{-1}\omega_1 \\ \Leftrightarrow &\quad x' \in \omega_1^{\perp} \end{aligned}$$

One thus has $([\pi]-\pi_1) \cdot \omega_1^{\perp} = \omega_2$, or equivalently $\omega_1^{\perp} = ([\pi]-\pi_1)^{-1}\omega_2$. The inclusion $\omega_1 \subseteq \omega$ then implies that $\omega \subseteq ([\pi]-\pi_1)^{-1}\omega_2$. In other words, $([\pi]-\pi_1) \cdot \omega \subseteq \omega_2$. □

2.2 Geometry of the special fiber

Let X be the special fiber of Y. Over X, the sheaf $\mathcal{E}$ is locally free of rank m over $O_X[\pi]/\pi^2$. The sheaves ω_1, ω_2 are in $\mathcal{E}[\pi]$, and contain $\pi \cdot \omega$.

Remark 2.5. The sheaf $\mathcal{E}[\pi]$ is totally isotropic, but is endowed with a perfect modified pairing given by

$$\{\pi x, \pi y\} :=< \pi x, y >$$

This pairing is **symmetric**; indeed since $\overline{\pi} = -\pi$ in the residue field of F, one has

$$\{\pi y, \pi x\} =< \pi y, x >=< y, \overline{\pi} x >=< \pi x, y >= \{\pi x, \pi y\}.$$

If we want to denote the orthogonal of a subspace $\mathcal{F} \subset \mathcal{E}[\pi]$ for this new pairing, we denote it by $\mathcal{F}^{\perp'}$ to highlight the difference with the usual pairing $<,>$, where we use the notation $\mathcal{F}^{\perp}$.

Definition 2.6. Let k be a field in characteristic p, and let $x \in X(k)$. Let us define the integers $(h(x), l(x))$ as the dimension of $\pi \cdot \omega$, and $\omega_1 \cap \omega_2$ respectively.

Remark 2.7. From the previous section, one gets that ω_2 is the orthogonal of ω_1 in $\mathcal{E}[\pi]$, for the modified pairing.

Proposition 2.8. *Let k be a field of characteristic p, and let $x \in X(k)$. Then one has*

$$0 \leq h(x) \leq l(x) \leq a$$

The integers $h(x), l(x)$ will allow us to define a stratification on $\overline{X}$. Indeed, one has

$$X = \coprod_{0 \leq h \leq l \leq a} X_{h,l}$$

where $X_{h,l}$ consists in the points x with $(h(x), l(x)) = (h, l)$.

Proposition 2.9. *Let (h, l) be integers with $0 \leq h \leq l \leq a$, and let $\overline{X_{h,l}}$ be the closure of $X_{h,l}$. Then*

$$\overline{X_{h,l}} \subseteq \coprod_{0 \leq h' \leq h \leq l \leq l' \leq a} X_{h',l'}$$

Proof. The integer h is equal to the dimension of $\pi \cdot \omega$. It thus decreases by specialization. The integer l is equal to the dimension of $\omega_1 \cap \omega_2$. This quantity increases by specialization. □

In particular, the stratum $X_{0,a}$ is closed and the strata $X_{h,h}$ are open, for every $0 \leq h \leq a$.

2.3 A remark on deformations of a p-divisible group

Let G be a p-divisible group over k, a field of characteristic p. Let $R = k[[t]]$ and $R_n = k[t]/(t^n)$, with the obvious maps. Let $\mathbb{D}$ the crystal of G and $\mathcal{E} = \mathbb{D}_{k \longrightarrow k}$. There are no divided powers on $k[[t]] \longrightarrow k$, thus we can't a priori evaluate $\mathbb{D}$ on $k[[t]]$. Let $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_k k[[t]]$. We denote $\omega \subset \mathcal{E}$ the Hodge filtration of G (with extra structure).

Proposition 2.10. *Let $\widetilde{\omega} \subset \widetilde{\mathcal{E}}$ be a locally direct factor lifting $\omega \subset \mathcal{E}$. Then there exists a p-divisible group $\widetilde{G}$ over $k[[t]]$ lifting G (with extra-structure), such that, when evaluating $\mathbb{D}(\widetilde{G})_{k[[t]] \longrightarrow k[[t]]} = \widetilde{\mathcal{E}}$, and the Hodge filtration is given by $\widetilde{\omega}$.*

Proof. All displays are Dieudonne displays (which we can consider since our ring is $R_n = k[t]/(t^n)$ or $R = k[[t]]$). We also denote $W(R)$ instead of $\hat{W}(R)$. Let P be the Display of G, and set $\widetilde{P} = P \otimes_{W(k)} W(R)$ and $P_n = P \otimes_{W(k)} W(k[t]/(t^n)) = \widetilde{P} \otimes_{W(k[[t]])} W(k[t]/(t^n))$. The map $k[[t]]/(t^n) \longrightarrow k[[t]]/(t^{n-1})$ is endowed with (nilpotent) divided powers. In particular, for $n = 2$, we get from $\widetilde{\omega}\otimes_{k[[t]]} k[t]/(t^2) \subset \widetilde{E}\otimes_{k[[t]]} k[t]/(t^2)$ a lift of the Hodge filtration which induces the existence of a display $\mathcal{P}_2$ with $W(R_2)$-module P_2, itself corresponding to a p-divisible group G_2 over R_2 lifting G, with $\mathbb{D}(G_2)_{R_2 \longrightarrow R_2} = \mathbb{D}(G)_{R_2 \longrightarrow k} = P_2$ and corresponding Hodge filtration (cf [Zin01] Theorem 4,[Mes72]). Assume the corresponding result at rank n. In particular we have a display $\mathcal{P}_n$, with module P_n and Hodge filtration corresponding to $\widetilde{w} \otimes_R R_n \subset P_n/I_{R_n}P_n = \widetilde{\mathcal{E}} \otimes_R R_n$. As the map corresponding to $R_{n+1} \longrightarrow R_n$ has divided powers and we have a R_{n+1}-triple (by base change ?), the lift of the Hodge filtration $\widetilde{w} \otimes R_{n+1} \subset \widetilde{\mathcal{E}} \otimes R_{n+1} = P_{n+1}/I_{R_{n+1}}P_{n+1}$ induces a lift $\mathcal{P}_{n+1}$ of $\mathcal{P}_n$, with Hodge filtration determined by $\widetilde{\omega}$. To $\mathcal{P}_{n+1}$ by [Zin01] Theorem 20 we have an associated p-divisible group G_{n+1} over R_{n+1}. Moreover, by [Lau14] Theorem B, we have $\mathbb{D}(G_{n+1})_{k[t]/(t^{n+1}) \longrightarrow k[t]/(t^{n+1})} = \mathbb{D}(G)_{k[[t]]/(t^n) \longrightarrow k[[t]]/(t^n)} = \mathbb{D}(\mathcal{P}_{n+1})_{k[[t]]/(t^n)} = \mathcal{P}_{n+1}/I_{R_{n+1}}P_{n+1} = \widetilde{\mathcal{E}} \otimes_R R_{n+1}$ with compatibility with the Hodge filtration. We then set $G = \varprojlim G_n$, a p-divisible group over $k[[t]]$, satisfying the desired assumptions. □

Using this proposition, by abuse we call $\widetilde{\mathcal{E}}$ the evaluation of $\mathcal{D}(G)$ on $k[[t]]$.

2.4 Closure relations and geometry of strata

In this section, we assume that $p \neq 2$. Let us prove a first proposition about the closure relations for the strata.

Proposition 2.11. *Let (h, l) be integers with $0 \leq h \leq l \leq a$. One has*

$$\overline{X_{h,l}} = \coprod_{0 \leq h' \leq h \leq l \leq l' \leq a} X_{h',l'}.$$

Proof. The previous proposition gives the expected inclusion, we will now show the converse. Let us first prove that $X_{0,a}$ is in the closure of $X_{h,l}$ for every $0 \leq h \leq l \leq a$. Let k be an algebraically closed field of characteristic p, and let $x \in X_{0,a}(k)$. We will prove that for any $h \leq l$, one can find a generization of x which lies in $X_{h,l}$.
Since $(h(x), l(x)) = (0, a)$, one has the inclusions $\omega_1 \subseteq \omega_2 \subseteq \omega = \mathcal{E}[\pi]$. The matrix of the modified pairing on this set, with some appropriate basis, is

$$\begin{pmatrix} 0 & 0 & I_a \\ 0 & I_{b-a} & 0 \\ I_a & 0 & 0 \end{pmatrix}$$

Let $\widetilde{\mathcal{E}}$ be the crystal evaluated at $k[[t]]$. We will investigate lifts of the modules $\omega_1 \subseteq \omega$ to $\widetilde{\mathcal{E}}$. First, a lift $\widetilde{\omega_1}$ of ω_1 inside $\widetilde{\mathcal{E}}[\pi]$ is given by a matrix $\begin{pmatrix} I_a \\ X \\ Y \end{pmatrix}$ where X, Y are matrices with coefficients in $k[[t]]$, whose reductions are 0 modulo t. One computes that the orthogonal of $\widetilde{\omega_1}$ inside $\widetilde{\mathcal{E}}[\pi]$ is given by $\begin{pmatrix} I_a & 0 \\ 0 & I_{b-a} \\ -{}^tY & -{}^tX \end{pmatrix}$. A lift $\widetilde{\omega}$ of ω will thus contain the vectors $\begin{pmatrix} 0 \\ I_{b-a} \\ -{}^tX \end{pmatrix}$, and be contained in

$\pi^{-1}\widetilde{\omega_1}$. Let us call $\pi\widetilde{e_1},\dots,\pi\widetilde{e_a}$ the vectors appearing in the matrix for $\widetilde{\omega_1}$, and choose $\widetilde{e_1},\dots,\widetilde{e_a}$ some preimage by π in $\widetilde{\mathcal{E}}$. Up to change $\widetilde{e_i}$ by some π-torsion element we can assume that they are two by two orthogonal for $<,>$.Considering the vectors $\pi e_{b+1},\dots,\pi e_{a+b},\widetilde{e_1},\dots,\widetilde{e_a}$, the remaining vectors for $\widetilde{\omega}$ are given by a matrix $\begin{pmatrix} I_a \\ Z \end{pmatrix}$ in the previous family, where Z is a matrix with coefficients in $k[[t]]$ whose reduction is 0. The condition that $\widetilde{\omega}$ is totally isotropic is equivalent to the equations

$$Z={}^tZ \qquad (Y+{}^tY+{}^tXX)Z=0.$$

Indeed, we have that the image by π of the element corresponding to $\begin{pmatrix} I_a \\ Z \end{pmatrix}$ are given by Z in the basis $\pi\widetilde{e_1},\dots\pi\widetilde{e_a}$ i.e. in the original basis $\pi e_1,\dots,\ \pi e_n$ by $\begin{pmatrix} Z \\ X^tZ \\ Y^tZ \end{pmatrix}$. The computation of $<w_1,w_2>$ for $w_1,w_2\in Z(\widetilde{e_1},\dots\widetilde{e_a})+\pi(e_{b+1},\dots,e_h)$ (written symbolicaly) is given by

$$<w_1,w_2>=<Z\sum\lambda_i\tilde{e}_i+\sum\lambda_i\pi e_{b+i},Z\ \sum\mu_i\tilde{e}_i+\sum\mu_i\pi e_{b+i})>$$
$$=<Z\sum\lambda_i\tilde{e}_i,Z\ \sum\mu_i\tilde{e}_i>+<Z\sum\lambda_i\tilde{e}_i,\sum\mu_i\pi e_{b+i}>+<\sum\lambda_i\pi e_{b+i},Z\ \sum\mu_i\tilde{e}_i>,$$

as $\mathcal{E}[\pi]$ is its own orthogonal, and the first term is zero as the $\widetilde{e}_i$ are two by two isotropic. Thus we are left with

$$<w_1,w_2>=-\{Z\sum\lambda_i\pi\tilde{e}_i,\sum\mu_i\pi e_{b+i}\}+\{\sum\lambda_i\pi e_{b+i},Z\ \sum\mu_i\pi\tilde{e}_i\},$$

which is given in matrix terms, varying w_1,w_2 and using the shape of the divided pairing by $-{}^tZ+Z=0$. The second equation is similar using orthogonality between $\widetilde{w}_1$ and the vector corresponding to $\begin{pmatrix} Z \\ XZ \\ YZ \end{pmatrix}$. The last equation is actually automatic. From Grothendieck-Messing, applied to a devissage to square zero ideals corresponding to $k[T]/(T^n)\longrightarrow k[T]/(T^{n-1})$, and Serre-Tate, this deformation of the Hodge filtration gives a generization $\widetilde{x}$ of x. We will use this argument repetitively. One then sees that $h(\widetilde{x})$ is equal to the rank of Z, and $l(\widetilde{x})$ is equal to the nullity of the matrix $Y+{}^tY+{}^tXX$. For any couple (l,h) with $0\leq h\leq l\leq a$, one can find matrices X,Y,Z such that the rank of Z is h, the rank of $Y+{}^tY+{}^tXX$ is $a-l$ and the above equations are satisfied, hence the result.
The general case can be deduced by a similar discussion : choose $h<\ell$ and $e_1,\dots,e_{a+b}$ a basis of $\mathcal{E}[\pi]$ such that $e_1,\dots,e_h$ is a basis of $\pi\omega$, $e_1,\dots,e_\ell$ a basis of $\omega^1\cap\omega^2$, $e_1,\dots,e_a$ is a basis of ω^1, $e_1,\dots,e_\ell,e_{a+1},\dots,e_{b+a-\ell}$ a basis of ω^2, and $e_1,\dots,e_{a+b-h}$ is a basis of $\omega[\pi]$. We moreover assume that the divided pairing is given by the matrix (in this basis)

$$\begin{pmatrix} & & & & & I_h \\ & & & & I_{\ell-h} & \\ & & I_{a-\ell} & 0 & & \\ & & 0 & I_{b-\ell} & & \\ & I_{\ell-h} & & & & \\ I_h & & & & & \end{pmatrix}$$

We then set in $\widetilde{\mathcal{E}}[\pi]$ a lift $\widetilde{\omega}_0$ of $\pi\omega$, $\widetilde{\omega}_{int} \supset \widetilde{\omega}_0$ of $\omega^1 \cap \omega^2$, and $\widetilde{\omega_1} \subset \widetilde{\omega}_{int}$ of ω_1 by the column of the matrix

$$\begin{pmatrix} I_h & & \\ & I_{\ell-h} & \\ & Y_2 & I_{a-\ell} \\ & & \\ & 0 & \\ & & \end{pmatrix}$$

for some matrix $Y_2 \in M_{a-\ell,\ell-h}(tk[[t]])$. Denote $\tilde{e}_1,\dots,\tilde{e}_h = e_1,\dots,e_h$ the first vectors in the previous matrix, which gives a basis for $\widetilde{\omega}_0$ and denote $\tilde{e}_{h+1},\dots,\tilde{e}_\ell$ the next $\ell-h$ ones. We then check that the orthogonal of $\widetilde{\omega}_1$ for the divided pairing in $\widetilde{E}[\pi]$ is given by

$$\begin{pmatrix} I_h & & \\ & I_{\ell-h} & \\ & & 0 \\ & & I_{b-\ell} \\ & 0 & \\ & & \end{pmatrix}$$

and this defines a lift $\widetilde{\omega}_2$ of ω_2 and, after inverting t, the intersection of $\widetilde{\omega}_2 \cap \widetilde{\omega}_1$ contains $\widetilde{\omega}_0$ and a space related to the kernel of the matrix Y_2. Namely, $\dim \widetilde{\omega}_2 \cap \widetilde{\omega}_1[1/t] = h + \dim \ker Y_2$. Finally we lift ω by adding to $\widetilde{\omega}_1$ the vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ I_{b-\ell} \\ 0 \\ 0 \end{pmatrix}$$

of $\widetilde{E}[\pi]$ together with the following vectors : choose $\pi^{-1}\tilde{e}_1,\dots,\pi^{-1}\tilde{e}_d$ a family of vectors of ω which generates $\omega[\pi]$ after multiplying by π, and choose lifts in $\widetilde{E}$ which moreover maps to $\tilde{e}_1,\dots,\tilde{e}_h = e_1,\dots,e_h$ (the previous basis for $\widetilde{\omega}_0$) and choose $\pi^{-1}\tilde{e}_{h+1},\dots,\pi^{-1}\tilde{e}_\ell$ in the preimage by π of $\tilde{e}_{h+1},\dots,\tilde{e}_\ell$ such that $\pi^{-1}\tilde{e}_1,\dots,\pi^{-1}\tilde{e}_\ell$ are two by two orthogonal for the original pairing $<,>$. Then we add $\pi^{-1}\tilde{e}_1,\dots,\pi^{-1}\tilde{e}_h$ and the vectors given in the basis $e_{b+a-\ell+1},\dots,e_{b+a-d},\pi^{-1}\tilde{e}_{d+1},\dots,\pi^{-1}\tilde{e}_\ell$ by the matrix $\begin{pmatrix} I_{\ell-d} \\ Z \end{pmatrix}$ i.e. in the original basis $\pi^{-1}e_1,\dots,\pi^{-1}(e_{a+b})$ by

$$\begin{pmatrix} I_h & 0 \\ 0 & Z \\ 0 & Y_2Z \\ 0 & 0 \\ 0 & \pi I_{\ell-h} \\ 0 & 0 \end{pmatrix}$$

Because of the assumption on $\pi^{-1}\tilde{e}_1,\dots,\pi^{-1}\tilde{e}_\ell$ these last ℓ vectors are two by two isotropic for the original pairing iff (this reduces to a calculation with the divided pairing $\{,\}$) ${}^tZ - Z = 0$. Moreover $\widetilde{\omega}$ is totally isotropic (for $<,>$) if moreover $Y_2Z = 0$. But clearly the rank, after inverting t, of $\pi\widetilde{\omega}$ is $h + \mathrm{rk}\, Z$, and thus if we have $h \leq h+r \leq \ell' = h+s \leq \ell$ for some $s,r \geq 0$, with $\ell - h \geq s \geq r$

then we can choose a symmetric matrix Z of rank r and a matrix Y_2 with kernel of dimension $s \geq r$ such that $Y_2 Z = 0$. Concluding as in the case of $X_{0,a}$, we have the result. □

Proposition 2.12. *For all $h \leq \ell$, the stratum $X_{h,l}$ is nonempty and smooth, and is equidimensional of dimension $ab - \frac{(l-h)(l-h+1)}{2}$.*

Proof. To prove that all the strata are non empty, by the proposition above, it is enough to prove that $X_{0,a}$ is non empty. Let $k = \overline{\mathbb{F}_p}$, and let E be an elliptic curve over k with complex multiplication by O_F such that the action of O_F on ω_E is given by σ_1, and let E^c be the same elliptic curve, but with the action of O_F twisted by the complex conjugation. Define $A = E^a \times (E^c)^b$, and let us choose a space $\omega_1 \subseteq \omega_A$ which is totally isotropic for the modified pairing. This gives a point in $X_{0,a}$. Let us now compute the dimension of the stratum $X_{h,l}$. On this stratum, on has the sheaves

$$\pi\omega \subseteq \omega_1 \cap \omega_2 \subseteq \omega_1$$

which are locally free of rank h, l, a respectively. Deforming a point of $X_{h,l}$ inside $X_{h,l}$ thus consists in the following operations:

- deforming the sheaf $\pi\omega$ in a sheaf $\widetilde{\omega_0}$, which should be totally isotropic for the modified pairing.
- deforming the sheaf $\omega_1 \cap \omega_2$ inside the orthogonal of the previous one, which should also be totally isotropic (for the modified pairing).
- deforming the sheaf ω_1 inside the orthogonal of the previous one (for the modified pairing), asking moreover that $\omega_1/(\omega_1 \cap \omega_2) \cap (\omega_1/(\omega_1 \cap \omega_2))^{\perp''} = \{0\}$, where $\perp''$ is the modified pairing descended to $(\omega_1 \cap \omega_2)^{\perp'}/(\omega_1 \cap \omega_2)$.
- deforming ω, which should contain the orthogonal of $\widetilde{\omega_0}$ (for the modified pairing) and be contained in $\pi^{-1}\widetilde{\omega_0}$, and be totally isotropic (for the original pairing). Note that[1] if we denote $\mathcal{F} = (\widetilde{\omega}_0)^{\perp'}$ we have $\mathcal{F} \subset \mathcal{E}[\pi]$ thus $\pi\mathcal{F}^\perp = \mathcal{F}^{\perp'} = \widetilde{\omega}_0$. But both $\mathcal{F}^\perp$ and $\pi^{-1}\widetilde{w}_0$ contains $\mathcal{E}[\pi]$, and are equal after multiplying by π, thus $\mathcal{F}^\perp = \pi^{-1}\widetilde{\omega}_0$. It is thus enough to deform the image of ω in $\mathcal{F}^\perp/\mathcal{F}$.

In other words we look at the following sequence of schemes

$$\mathrm{Gr}^{Sp}(h, (\mathcal{F}^\perp/\mathcal{F}, <,>)) \longrightarrow U \longrightarrow \mathrm{Gr}^O(l-h, ((\omega_0)^{univ,\perp'}/\omega_0^{univ}, \{,\})) \longrightarrow \mathrm{Gr}^O(h, (\mathcal{E}[\pi], \{,\})),$$

where $\mathrm{Gr}^O(k, (V, \{,\}))$ is the Grassmanian of totally isotropic subspace of rank k in a space V with symmetric pairing, and $\mathrm{Gr}^{Sp}(k, (V, <,>)))$ is the analogous one for an alternated pairing, $(\omega_0)^{univ}$ is the universal object of $\mathrm{Gr}^O(h, ((\mathcal{E}[\pi], \{,\}))$, and $U \subset \mathrm{Gr}^O(a-l, (\omega_1 \cap \omega_2)^{univ,\perp'}/(\omega_1 \cap \omega_2^{univ}), \{,\})$ is the open where the universal isotropic subspace F, which corresponds to $\omega_1/(\omega_1 \cap \omega_2)$, satisfies $F^{\perp'} \cap F = \{0\}$, with $\omega_1 \cap \omega_2^{univ}$ the (pullback of the) universal object of $\mathrm{Gr}^O(l-h, ((\omega_0)^{univ}, \{,\}))$.

All those Grassmanians are relatively smooth, and the first point gives a dimension $h(a+b-h) - \frac{h(h+1)}{2}$, the second one a relative dimension $(l-h)(a+b-l-h) - \frac{(l-h)(l-h+1)}{2}$, the third one $(a-l)(b-l)$ and the last one $\frac{h(h+1)}{2}$. The total dimension is then

$$ab - \frac{(l-h)(l-h+1)}{2}.$$

[1] $\pi\mathcal{F}^\perp = \mathcal{F}^{\perp'}$ if $\mathcal{F} \subset \mathcal{E}[\pi]$ and $\mathcal{G}^\perp = (\pi\mathcal{G})^{\perp'}$ if $\mathcal{E}[\pi] \subset \mathcal{G}$

Moreover if $x \in X_{h,l}(S)$ is a R-point for some ring R, and $S \longrightarrow R$ is a square-zero thickening of $\mathbb{F}_p$-schemes, then to lift x it is enough to lift its Hodge Filtration (by Grothendieck Messing) satisfying all the desired properties. But this is indeed possible as all the previous Grassmanians are formally smooth and the dimension is indeed the given one.

□

Corollary 2.13. *The irreducible components of the scheme X are determined by the ones of $X_{h,h}$ for $0 \leq h \leq a$.*

2.5 Local rings

Assume $p \neq 2$.

Proposition 2.14. *The smooth locus of X is the union of the strata $X_{h,h}$ for $0 \leq h \leq a$.*

Proof. We have seen that the strata $X_{h,h}$ are open and smooth in the previous proposition, thus their union is included in the smooth locus. Assume now that $h < l$, and let $x \in X_{h,l}(k)$. We will prove that X is not smooth at x. The pairing on $\mathcal{E}[\pi]$ is given by a matrix

$$\begin{pmatrix} 0 & 0 & I_l \\ 0 & I_{a+b-2l} & 0 \\ I_l & 0 & 0 \end{pmatrix}$$

written with respect to a basis $\pi e_1, \ldots, \pi e_{a+b}$, where ω_1 is spanned by $\pi e_1, \ldots, \pi e_a$, $\omega_1 \cap \omega_2$ is spanned by $\pi e_1, \ldots, \pi e_l$ and $\pi\omega$ is spanned by $\pi e_{l-h+1}, \ldots, \pi e_l$. One can also assume that ω/ω_1 is spanned by $\pi e_{a+1}, \ldots, \pi e_{a+b-h}, e_{l-h+1}, \ldots, e_l$. Let $\mathcal{E}'$ be the evaluation of the crystal at $k[\varepsilon]/\varepsilon^2$. Let us define a lift ω' of ω to $\mathcal{E}'$. One can lift the basis on $\mathcal{E}$ to a basis $e'_1, \ldots, e'_{a+b}$ in such a way that the matrix of the pairing is not changed. We define ω'_1 to be spanned by $\pi e'_1 + \varepsilon\pi e'_{a+b-l+1}, \pi e'_2, \ldots, \pi e'_a$. We thus define ω'/ω'_1 to be spanned by $\pi e'_{a+1}, \ldots, \pi e'_{a+b-l+1} + \varepsilon e'_1, \ldots, \pi e'_{a+b-h}, e'_{l-h+1}, \ldots, e'_l$. This gives a point $x' \in X(k[\varepsilon]/\varepsilon^2)$. We will now prove that this point cannot be lifted to $k[\varepsilon]/\varepsilon^3$. If it were the case, one would have a lift $\widetilde{\mathcal{E}}$ of $\mathcal{E}'$ to $k[\varepsilon]/\varepsilon^3$ together with a lift $\widetilde{\omega}$ of ω'. In particular, there would exist an element in $\widetilde{\omega_1}$ of the form $v_1 := \pi\widetilde{e_1} + \varepsilon\pi\widetilde{e_{a+b-l+1}} + \varepsilon^2\pi u$. There would also be in $\widetilde{\omega}$ an element of the form $v_2 := \pi\widetilde{e_{a+b-l+1}} + \varepsilon(\widetilde{e_1} + \varepsilon\widetilde{e_{a+b-l+1}}) + \varepsilon^2 v$ such that $\varepsilon^2\pi v$ belongs to $\widetilde{\omega_1}$. The (original) pairing between these two vectors is equal to

$$< v_1, v_2 >= \{v_1, \pi v_2\} = 2\varepsilon^2 + \{\varepsilon^2\pi\widetilde{e_1}, \pi v\}.$$

But the quantity $\{\varepsilon^2\pi\widetilde{e_1}, \pi v\} = 0$ modulo ε^3, since $\varepsilon^2\pi v$ belongs to $\widetilde{\omega_1}$. This gives the desired contradiction, since $\widetilde{\omega} \ni v_1, v_2$ should be totally isotropic. □

Proposition 2.15. *The scheme Y is flat over $\mathcal{O}_F$, normal and its special fiber is a complete intersection (thus Cohen-Macaulay).*

Proof. The special fiber of Y, X, is reduced as its irreducible components are smooth. Thus we only need to prove that generic points of irreducible components of X lifts to characteristic zero. We translate the formulation this way. Let $\Lambda = \mathcal{O}_F^{2m}$, with natural polarisation $(x, y) = \sum_i x_i c(y_i)$. We have a local model diagram

$$Y \longleftarrow \widetilde{Y} \longrightarrow \mathcal{N},$$

where $\widetilde{Y} = \mathrm{Isom}_{\mathcal{O}_F,<,>}(\mathcal{E}, \Lambda \otimes O_Y)$, and $\mathcal{N}$ is the $\mathcal{O}_{F,p}$-scheme parametrizing, for any scheme S, triples

$$(\mathcal{F}_1, \mathcal{F}),$$

where $\mathcal{F} \subset \Lambda \otimes_{O_F} S$ is a locally direct factor, stable by O_F, totally isotropic of rank $a+b$, and $\mathcal{F}_1 \subset \mathcal{F}$ is a locally direct factor of rank a, stable by $\mathcal{O}_F$, such that

- $([\pi] - \pi_1)(\mathcal{F}_1) = 0$
- $([\pi] - \pi_2)(\mathcal{F}) \subset \mathcal{F}_1$.

where the map $\widetilde{Y} \longrightarrow Y$ is the natural one and is smooth (it is a torsor over a smooth algebraic group), and

$$\widetilde{Y} \longrightarrow \mathcal{N}, (A/S, \omega^1, i : \mathcal{E} \simeq \Lambda \otimes \mathcal{O}_S) \mapsto \omega^1 \subset \omega \subset \mathcal{E} \simeq \Lambda \otimes \mathcal{O}_S.$$

In particular $\widetilde{Y} \longrightarrow \mathcal{N}$ is formally smooth by Grothendieck-Messing. Thus to prove the result, it is enough to show that $\mathcal{N}$ is flat over $O_{F,p}$. But what we did before actually shows that, if $N = \mathcal{N} \otimes_{O_F} k_F$,

$$N = \coprod_{0 \leq h \leq \ell \leq a} N_{h,l},$$

with the expected closure relation, and each $N_{h,l}$ is smooth of some dimension. In particular, N is reduced and the irreducible components of N are those of the various $N_{h,h}$. In particular, N has dimension ab and $\mathcal{N}$ has dimension $ab+1 = \dim N + \dim O_F$. Thus to show that $\mathcal{N}$ is flat over O_F it is enough to show it is Cohen-Macaulay, by the Miracle flatness theorem. Moreover, the non-CM locus is closed, and $\mathcal{N}$ is smooth in generic fiber thus contains concentrated in the special fiber. But N is endowed with an action of

$$G = \{g \in \mathrm{GL}_{a+b,k[X]/(X^2)}(\Lambda) | < gx, y >=< x, gy >\},$$

and G acts transitively on $X_{h,\ell}$ for each h, ℓ : Indeed, for each such we can find a basis $e_1, \dots, e_{a+b}$ of Λ such that $e_1, \dots, e_h, Xe_1, \dots, Xe_{a+b-h}$ is a basis of $\mathcal{F}$ and $Xe_1, \dots, Xe_a$ is a basis of $\mathcal{F}^1$, with $(F^1) \cap (F^1)^\perp$ given by $Xe_1, \dots, Xe_\ell$. But G thus preserves the non-CM locus and if this is non empty there exists a point $x \in X_{0,a}$ in the non-CM locus. But we have calculated the completed local ring at x, it is given by

$$k(x)[Z, X, Y]/(Z - {}^tZ, (Y + {}^tY + {}^tXX)Z),$$

where Z is a (symmetric) $a \times a$-matrix, Y is a $a \times a$-matrix and X is a $b - a \times a$-matrix. In particular we have $a^2 + (b-a)a + a(a+1)/2$ variables and $a(a+1)/2$ relations, and N is of dimension ab. Thus, x is in the complete intersection locus, thus CM. Thus N and thus $\mathcal{N}$ is Cohen-Macaulay and $\mathcal{N}$ is flat over $O_{F,p}$ by Miracle flatness. Moreover $\mathcal{N}$ is smooth in generic fiber, and N is generically regular as $X_{h,h}$ is smooth for all h, thus $\mathcal{N}$ is R1 and S2, thus normal by Serre's criterion. The same is true for Y using the local model diagram. □

3 Modular forms

3.1 Sheaves

We define $\mathcal{E}_i = \mathcal{E}[T - \pi_i]$, for $i = 1, 2$.

Proposition 3.1. *The sheaf* $\det \mathcal{E}$ *is trivial, and one has* $\det(\mathcal{E}_1) \simeq \det(\mathcal{E}_2)^{-1}$.

Proof. The follows from the fact that $\mathcal{E}$ has an alternate pairing, that $\mathcal{E}_1$ is totally isotropic, and that the multiplication by $T-\pi_1$ induces an isomorphism $\mathcal{E}/\mathcal{E}_1 \simeq \mathcal{E}_2$. Indeed, $\det \mathcal{E} = \det \mathcal{E}_1 \otimes \det(\mathcal{E}/\mathcal{E}_1)$, and the map

$$\mathcal{E}_1 \longrightarrow \mathcal{E} \xrightarrow[<,>]{\sim} \mathcal{E}^\vee \longrightarrow \mathcal{E}_1^\vee,$$

is zero as $\mathcal{E}_1$ is totally isotropic. We thus deduce an isomorphism $\mathcal{E}_1 \xrightarrow{\sim} (\mathcal{E}/\mathcal{E}_1)^\vee$, and finally

$$\det \mathcal{E} = \det \mathcal{E}_1 \otimes \det(\mathcal{E}_1)^{-1} = \mathcal{O}_S.$$

□

Proposition 3.2. *One has isomorphisms*

$$\det(\mathcal{E}_1) \simeq \det(\omega_1) \otimes \det(\omega_2)^{-1} \simeq \det(\omega/\omega_2) \otimes \det(\omega/\omega_1)^{-1}$$

Proof. Inside $\mathcal{E}$ the orthogonal of ω_1 is $(T-\pi_1)^{-1}\omega_2$. This implies that $\det((T-\pi_1)^{-1}\omega_2/\mathcal{E}_1) \simeq \det(\mathcal{E}_1/\omega_1)^{-1} \simeq \det(\mathcal{E}_1)^{-1} \otimes \det(\omega_1)$. Moreover, the multiplication by $T-\pi_1$ induces an isomorphism between $(T-\pi_1)^{-1}\omega_2/\mathcal{E}_1$ and ω_2.
For the second part, one uses $\det \omega = \det \omega_1 \otimes \det \omega/\omega_1 = \det \omega_2 \otimes \det \omega/\omega_2$. □

Proposition 3.3. *On the Rapoport locus, one has an isomorphism* $\det(\omega/\omega_2) \simeq \det(\omega_1)$.
In the special fiber, one has an isomorphism $\mathcal{E}_1 \simeq \mathcal{E}_2$. *The sheaf* $\varepsilon := \det \mathcal{E}_1$ *satisfies* $\varepsilon^2 \simeq \mathcal{O}_S$.

3.2 Definition and vanishing modulo p

As we work in characteristic p, we will need to use an integral version of Schur functors. See also [Gol14] section 3.8. For M a rank r free module over R, choose an isomorphism $M \simeq R^r$, and denote $\mathcal{L}(\lambda)$ the sheaf on GL_r/B (B the upper triangular Borel of GL_r) whose sections are given by

$$\mathcal{L}(\lambda)(U) = \{f : \pi^{-1}(U) \longrightarrow \mathbb{A}^1 | f(gb) = \lambda^{-1}(b)(g) \forall b \in B, g \in \pi^{-1}(U)\}.$$

Denote by $\mathcal{L}_M(\lambda)$ the sheaf on the flag variety $\mathcal{F}\ell(M)$ for M, given by $\phi_*\mathcal{L}(\lambda)$ after choosing an isomorphism $\phi : R^r \simeq M$ (inducing $\mathrm{GL}_r \simeq \mathrm{Isom}_R(R^r, M)$ and $\phi : \mathrm{GL}_r/B \simeq \mathcal{F}\ell(M)$). This is independant of the choice of ϕ. For $\underline{a} = (a_1 \geq \cdots \geq a_r) \in \mathbb{Z}^r$, with associated character of $T = \mathbb{G}_m^r \subset B$, denote $M^{(a_1,\dots,a_r)}$ the global sections of $\mathcal{L}_M(\underline{a})$, i.e.

$$M^{(a_1,\dots,a_r)} = H^0(\mathcal{F}\ell(M), \mathcal{L}_M(\underline{a})).$$

As $H^1(\mathcal{F}\ell(M), \mathcal{L}_M(\underline{a})) = 0$ (Kempf theorem, see [Jan03] Proposition 4.5), the formation of $M^{(a_1,\dots,a_r)}$ commutes with base change $R \longrightarrow R'$, and thus the construction glues to a functor from the category of rank r vector bundles on a scheme X to the category of vector bundles on X (of any rank) associating to $\mathcal{V}$ or rank r the vector bundle $\mathcal{V}^{(a_1,\dots,a_r)}$. We denote $\underline{a}^\vee = (-a_r, \dots, -a_1)$.

Definition 3.4. Let k, l, r be three integers. A (scalar-valued) modular form of weight (k, l, r) is a section of the sheaf

$$(\det \omega_1)^k \otimes (\det \omega/\omega_1)^l \otimes (\det \mathcal{E}_1)^r.$$

More generally, given $\underline{k} = (k_1, \dots, k_a) \in \mathbb{Z}^a, \underline{\ell} = (\ell_1, \dots, \ell_b) \in \mathbb{Z}^b$ with $k_1 \geq \cdots \geq k_a, \ell_1 \geq \cdots \geq \ell_b$, we can consider the sheaf

$$\omega^{(\underline{k},\underline{\ell},r)} := \omega_1^{\underline{k}} \otimes (\omega/\omega_1)^{\underline{\ell}} \otimes (\det \mathcal{E}_1)^r.$$

A weight $(\underline{k}, \underline{\ell}, r)$ modular form is a section of this sheaf.

Remark 3.5. In generic fiber we can remove the use of r, and we can replace ω/ω_1 by ω_2. In special fiber though, $\omega_1 = \omega_2$ up to a square zero sheaf. In special fiber, we can moreover assume that $r = 0, 1$.

In special fiber we have the following vanishing result.

Proposition 3.6. *If $-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1$ is not decreasing (i.e. if $k_1 > k_a$ or $k_a > \ell_b$), then*

$$H^0(\overline{X_{0,0}}, \omega^{(\underline{k},\underline{\ell},r)}) = 0.$$

Proof. Let $x \in X_{0,0}$. Then above x we have $\omega_1 \subset \mathcal{E}[\pi] = \mathcal{E}_1 \ = \omega$. We look at $\mathrm{Gr}_{a,a+b}(\mathcal{E}[\pi])$ the Grassmanian of rank a sub-bundles of $\mathcal{E}[\pi]$. Over it, we have a universal bundle $V_1 \subset \mathcal{E}[\pi]$, which induces an immersion $\mathrm{Gr}_{a,a+b}(\mathcal{E}_x[\pi]) \longrightarrow \overline{X_{0,0}}$ mapping ω_1 to x. The pullback of $\omega = \mathcal{E}[\pi]$ to $\mathrm{Gr}_{a,a+b}$ is constant, and the pullback of the universal ω_1 on $\overline{X}$ is

$$V_1 =: \mathcal{O}(\underbrace{-1, 0, \dots, 0}_{a \text{ times}}, 0, \dots, 0)$$

(which corrresponds to $\mathcal{O}(-1)$) up to twist by center on $\mathbb{P}^1$). Thus, the pullback of ω/ω_1 is

$$\mathcal{E}[\pi]/V_1 =: \mathcal{O}(0, \dots, 0, \underbrace{0, \dots, 0, -1}_{b \text{ times}}),$$

(which corresponds to $\mathcal{O}(1)$ up to twist by the center on $\mathbb{P}^1$). The restriction of a section of $\omega^{\underline{k},\underline{\ell},r}$ to $\mathrm{Gr}_{a,a+b}$ is then $\mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee)$. Remark that $\mathrm{Gr}_{a,a+b} \simeq P\backslash G$ for $G = \mathrm{GL}_{a+b}$ and P the standard parabolic of size a, b. We thus have a map $B\backslash G \xrightarrow{\pi} P\backslash G$ (for the upper triangular Borel B), and

$$\mathcal{L}_P(\underline{k}, \underline{\ell}) = \pi_* \mathcal{L}(-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1),$$

with $\mathcal{L}(-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1)$ the line bundle on $B\backslash G$. But

$$H^0(P\backslash G, \mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee)) = H^0(B\backslash G, \mathcal{L}(-k_1, \dots, -k_a, -\ell_b, \dots, -\ell_1)) = 0$$

under the assumption (see next Proposition 3.7). This is true for all points of $X_{0,0}$ thus we have the vanishing result. ☐

The following is well known,

Proposition 3.7. *Let G be a split reductive group in characteristic p. Let $B \subset P \subset G$ be a Borel and a parabolic subgroup, and T a torus. Denote $\pi : G \longrightarrow G/B$ and $f : G/B \longrightarrow G/P$. Let $\lambda \in X(T)$ be a weight. Let $\mathcal{L}(\lambda)$ be the line bundle on G/B such that*

$$\mathcal{L}(\lambda)(U) = \{f : \pi^{-1}(U) \longrightarrow \mathbb{A}^1 | f(gb) = \lambda^{-1}(b)(g) \forall b \in B, g \in \pi^{-1}(U)\},$$

and $\mathcal{L}_P(\lambda) = f_*\mathcal{L}(\lambda)$. Then λ is dominant if and only if

$$H^0(G/P, \mathcal{L}_P(\lambda)) \neq 0.$$

Proof. See [Jan03] Section II.2 for the definitions. We have $H^0(G/P, \mathcal{L}_P(\lambda)) = H^0(G/B, \mathcal{L}(\lambda))$. But by [Jan03] Proposition 2.6, λ is dominant iff $H^0(G/B, \mathcal{L}(\lambda)) = 0$. ☐

Lemma 3.8. *Let $G = \mathrm{GL}_{a+b}$, and P a standard parabolic with Levi $\mathrm{GL}_a \times \mathrm{GL}_b$. Let $\mathcal{W}$ be the universal direct factor and $\mathcal{V}$ the universal quotient on $X = G/P$. Then $\mathcal{W}^{\underline{k}} \otimes \mathcal{V}^{\underline{\ell}}$ coincides with $\mathcal{L}_P(-\underline{k}, \underline{\ell}^\vee)$.*

Proof. In particular we need to prove that $\mathcal{V} = \mathcal{V}^{(1,0,\dots,0)} = \mathcal{L}_P(0,\dots,0,-1)$ and $\mathcal{W} = \mathcal{L}_P(-1,0,\dots,0)$. But conversely, as $\mathcal{L}_P$ is compatible with tensor product (on sheaves) and sum (on characters, see [Jan03] Chapter 4), and as Schur functors commutes with base change, it is enough to check this at the fiber over $1 \in G/P$ as a P representation. But clearly $\mathcal{L}_P(0,\dots,0,-1)^{\underline{\ell}} = \mathcal{L}_P(0,\dots,0,-\ell_b,\dots,-\ell_1)$ as P-representation and similarly for $\mathcal{L}_P(-1,0,\dots,0)$. To prove that $\mathcal{V}$ coincides with $\mathcal{L}_P(0,\dots,0,1)$, recall that both are G-equivariant vector bundles, so we can check the isomorphism at the fiber above $1 \in G/P$. But it is clear that there $< e_1,\dots,e_a >= \mathcal{W}_1 = V_P(-1,0,\dots,0)$ as P-representation. $\square$

Now let $x \in X_{h,h}$ for some $h \leq a$. We have

$$0 \subset \pi\omega_x \subset \omega_1 \subset \omega_x[\pi] \subset \mathcal{E}[\pi].$$

In particular ω_1 gives a point of $\mathrm{Gr}_{a-h,a+b-2h}(\omega_x[\pi]/\pi\omega_x)$, and there is a natural map

$$\mathrm{Gr}_{a-h,a+b-2h}(\omega_x[\pi]/\pi\omega_x) \longrightarrow \overline{X_{h,h}}.$$

Moreover, the pullback of $\pi\omega, \omega[\pi]$ to the Grassmanian is constant (by construction) and thus we have an extension

$$0 \longrightarrow \omega[\pi] = \mathcal{O}^h \longrightarrow \omega_1 \longrightarrow \omega_1/\omega[\pi] = \mathcal{O}(-1) \longrightarrow 0,$$

and

$$0 \longrightarrow \omega[\pi]/\omega_1 = \mathcal{O}(1) \longrightarrow \omega/\omega_1 \longrightarrow \omega/\omega[\pi] \simeq \pi\omega \longrightarrow 0,$$

thus we can use the previous strategy to prove the following.

Theorem 3.9. *Assume $h < a$. If we cannot find $a-h$ indexes $i_t \in \{1,\dots,a\}$ and $b-h$ indexes $j_s \in \{1,\dots,b\}$ such that*

$$k_{i_1} = \dots = k_{i_{a-h}} \leq \ell_{j_1} \leq \dots \leq \ell_{j_{b-h}},$$

then

$$H^0(\overline{X_{h,h}}, \omega^{(\underline{k},\underline{\ell},r)}) = 0.$$

Remark 3.10. Note that this is the case in particular if $\underline{k}$ is regular enough, or if $h+1$ weights of $\underline{k}$ are greater than $\underline{\ell}$. The most restrictive case to apply the theorem is when $h = a-1$, in which case we can apply it under the assumption $k_a > \ell_{b-a+1}$.

Proof. By what preceed, we can choose a point $x \in X_{h,h}$ and compute the global sections of $\omega^{(\underline{k},\underline{\ell},r)}$ on the associated Grassmanian $\mathrm{Gr}_x := \mathrm{Gr}_{a-h,a+b-2h}(\omega[\pi]/\omega)$ (seen as a closed subspace of $\overline{X_{h,h}}$). On this space, $\mathcal{E}$ is constant (the p-divisible group is fixed), thus we can forget about r. We denote the following subgroups of GL_{a+b-h} :

$$M = \begin{pmatrix} \mathrm{GL}_h & & 0 \\ & \mathrm{GL}_{a+b-2h} & \\ 0 & & \mathrm{GL}_h \end{pmatrix} \supset P = \begin{pmatrix} \mathrm{GL}_h & & 0 & \\ & \mathrm{GL}_{a-h} & \star & \\ & 0 & \mathrm{GL}_{b-h} & \\ 0 & & & \mathrm{GL}_h \end{pmatrix}$$

and

$$P_{a-h,b-h} = \begin{pmatrix} \mathrm{GL}_{a-h} & \star \\ 0 & \mathrm{GL}_{b-h} \end{pmatrix} \subset \mathrm{GL}_{a+b-2h}\,.$$

Clearly, we have an isomorphism $\mathrm{Gr}_{a-h,a+b-2h} := P_{a-h,b-h}\backslash \mathrm{GL}_{a+b-2h} \simeq P\backslash M =: \mathrm{Gr}$, and we will use the partial Borel-Weyl-Bott theorem on $P\backslash M = \mathrm{Gr}$. Denote V the vector space of dimension $a+b$ on which M acts, it corresponds to a vector bundle $\mathcal{V}$ on Gr, which coincides with the pullback of ω to Gr. As representation of M, $V = V_0 \oplus V_1 \oplus V_2$, a sum of irreducible, and we need to compute the weights of the representation $V^{\underline{k},\underline{\ell}}$ (the Schur functor for GL_{a+b} associated to $(\underline{k},\ \underline{\ell})$) for the action of P. But as a representation of GL_{a+b}, $V^{\underline{k},\underline{\ell}}$ has weights $w \cdot (k_1, \dots, k_a, \ell_1, \dots, \ell_b), w \in \mathfrak{S}_{a+b}$. Among those weights, the highest weights for the action of P are those of the form $w_1 w_2 \cdot (k_1, \dots, k_a, \ell_1, \dots, \ell_b)$ with $(w_1, w_2) \in \mathfrak{S}_a \times \mathfrak{S}_b$ and

$$w_1(1) \geq \dots \geq w_1(h), \quad w_1(h+1) \geq \dots \geq w_1(a), \quad w_2(1) \geq \dots \geq w_2(b-h),$$

$$w_2(b-h+1) \geq \dots \geq w_2(b).$$

Denote ${}^PW$ this space. Thus, $\mathcal{V}^{\underline{k},\underline{\ell}}$ (and thus $\omega^{(\underline{k},\underline{\ell})}$) is an extension of $\mathcal{L}_P(w_1 w_2 \cdot (-\underline{k}, \underline{\ell}^\vee))$ for $w \in {}^PW$. But under the hypothesis non of these bundles have sections (Proposition 3.7), thus $H^0(\mathrm{Gr}, V^{\underline{k},\underline{\ell}}) = H^0(\mathrm{Gr}_x, \omega^{\underline{k},\underline{\ell},r}) = 0$. As this is true for any point $x \in X_{h,h}$, we deduce the result. □

4 Further strata for the case $(1, n)$

In this section, we consider the case where $(a, b) = (1, n)$, where $n \geq 1$ is an integer.

4.1 Definition of the invariants

We will define some invariants on the special fiber X. Let us recall that one has locally free sheaves ω_1, ω_2, of rank respectively 1 and n.

Definition 4.1. We define $b \in H^0(X, (\omega/\omega_2) \otimes \omega_1^{-1})$ thanks to the natural inclusion $\omega_1 \to \omega/\omega_2$. We define $m \in H^0(X, \omega_1 \otimes (\omega/\omega_2)^{-1}$ thanks to the multiplication by $\pi : \omega/\omega_2 \to \omega_1$.
For $i \in \{1, 2\}$, we define $hasse_i \in H^0((\omega/\omega_i)^{(p)} \otimes \omega_1^{-1})$ thanks to the map $hasse : \mathcal{E}[\pi] \to (\omega/\omega_i)^{(p)}$, induced by the composition of the Verschiebung and the division by π.

We refer to [Bij16] Def. 3.8 for more details about the definition of the maps $hasse_i$ (note that the reference deal with the ordinary case i.e $a = b$).

Proposition 4.2. *We have the following properties.*

- *One has $bm = 0$ and $mb = 0$.*
- *If x is point of X with $b(x) = 0$, then $hasse_1(x) = 0$ implies that $hasse_2(x) = 0$.*
- *If x is point of X with $b(x) \neq 0$, then one cannot have $hasse_1(x) = 0$ and $hasse_2(x) = 0$.*

Remark 4.3. The stratification defined previously consists in three strata, according to whether the sections b and m are 0 or not.

Proof. Indeed clearly $mb = 0$ as $\pi\omega_1 = 0$. Moreover, as $\pi\omega \subset \omega_2$ (as $\omega_1 \subset (\pi\omega)^{\perp'}$ because $(\pi\omega)^{\perp'} = (\pi(\omega + \mathcal{E}[\pi]))^{\perp'} = (\omega + \mathcal{E}[\pi])^{\perp}$ and this last space contains ω_1 as both ω and $\mathcal{E}[\pi]$ are totally isotropic), we have clearly that $bm = 0$.

For the second point, if $b = 0$ then $\omega_1 \subset \omega_2$ and thus if $\text{hasse}_1 = 0$, i.e. $\text{hasse}(\omega_1) \subset \omega_1^{(p)}$ then $\text{hasse}(\omega_1) \subset \omega_2^{(p)}$. For the last point, remark that if x is a point, then $b \neq 0$ is equivalent to $\omega = \omega_1 \oplus \omega_2$ as ω_1 is of rank 1. Thus the vanishing of both hasse_1 and hasse_2 is equivalent to the vanishing of $\omega_1 \xrightarrow{\text{hasse}} \omega^{(p)}$. But because $\omega_1 \oplus \omega_2 = \omega$, which is thus of π-torsion, hasse, which is surjective, induces an isomorphism $\mathcal{E}[\pi] \xrightarrow{\text{hasse}} \omega^{(p)} = \mathcal{E}[\pi]^{(p)}$, and thus its restriction to ω_1 can't be zero. ◻

Let us now define the different strata that we will consider.

- The ordinary locus is $X^{ord} = \{x \in X, m(x) \neq 0, hasse_2(x) \neq 0\}$.
- $R_1 = \{x \in X, m(x) \neq 0, hasse_1(x) \neq 0, hasse_2(x) = 0\}$.
- $R_2 = \{x \in X, m(x) \neq 0, hasse_1(x) = 0\}$.
- $B_0 = \{x \in X, b(x) \neq 0, hasse_1(x) \neq 0, hasse_2(x) \neq 0\}$.
- $B_1 = \{x \in X, b(x) \neq 0, hasse_2(x) = 0\}$.
- $B_2 = \{x \in X, b(x) \neq 0, hasse_1(x) = 0\}$.
- $P_0 = \{x \in X, m(x) = b(x) = 0, hasse_2(x) \neq 0\}$.
- $P_1 = \{x \in X, m(x) = b(x) = 0, hasse_1(x) \neq 0, hasse_2(x) = 0\}$.
- $P_2 = \{x \in X, m(x) = b(x) = 0, hasse_1(x) = 0\}$.

Proposition 4.4. *Let x be a point in X^{ord}. Then x is μ-ordinary in the sense of [BH17]. In particular, one has $A[\pi] \simeq \mu_p \times \mathbb{Z}/p\mathbb{Z} \times LT^{n-1}$.*

Remark 4.5. Here LT is defined in [BH17] before Définition 1.1.3, this is X_β with $\beta = (1)$ ($e = 2$ and $\mathcal{T}$ is a singleton).

4.2 The conjugate filtration

The Verschiebung induces a map $V : \mathcal{E} \to \omega^{(p)}$, which is compatible with the action of π.

Definition 4.6. We define the sheave $\mathcal{F}_i$, $i = 1, 2$ by the formula

$$\mathcal{F}_i := \pi \cdot V^{-1}\omega_i^{(p)}$$

Proposition 4.7. *The sheaves $\mathcal{F}_i$, $i = 1, 2$ are locally free of rank 1 and n, and are included in $\mathcal{E}[\pi]$. Moreover, $\mathcal{F}_2$ is the orthogonal of $\mathcal{F}_1$ for the modified pairing.*

Proof. The sheaf $V^{-1}\omega_1^{(p)}$ is locally free of rank $n+2 = a+b+1$, and contains $\mathcal{E}[\pi]$. This implies that $\mathcal{F}_1$ is locally free of rank 1. One gets in a similar way the result for $\mathcal{F}_2$.
To prove the last part, one only needs to check that $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal. Let $x \in \mathcal{F}_1$ and $y \in \mathcal{F}_2$. By definition, there exist x', y' such that $x = \pi x'$ and $y = \pi y'$, and $Vx' \in \omega_1^{(p)}$, $Vy' \in \omega_2^{(p)}$. Since ω_1, and ω_2 are orthogonal for the modified pairing, one gets the relation $\{Vx', Vy'\} = 0$. The element Vx' is in $\mathcal{E}[\pi]^{(p)}$; there exists then $z \in \mathcal{E}^{(p)}$ such that $Vx' = \pi z$. Now one has

$$0 = \{Vx', Vy'\} = \{\pi z, Vy'\} =< z, Vy' >=< Fz, y' >$$

But there exists a unit u such that $uFz = \pi x' = x$, this equality being in $\mathcal{F}/\pi\mathcal{F}$, where $\mathcal{F} = KerV$. There exists then $a \in \mathcal{F}$ such that $Fz = u^{-1}x + \pi a$. Thus $0 =< u^{-1}x + \pi a, y' >= \{u^{-1}x, y\} - < a, y >= \{u^{-1}x, y\}$. Indeed, since a and y belong to $\mathcal{F}$, which is totally isotropic, one must have $< a, y >= 0$. One then observes that the quantity $\{u^{-1}x, y\}$ only depends on the class of u in O_F/π, and one concludes that $\{x, y\} = 0$. □

Proposition 4.8. *Let x be a point of X. Then the condition $hasse_2(x) = 0$ is equivalent to $\omega_1 \subseteq \mathcal{F}_2$. The condition $hasse_1(x) = 0$ is equivalent to $\omega_1 = \mathcal{F}_1$.*

4.3 Stratification when $n > 1$

First, we remark that R_1 and P_1 are empty if $n \leq 2$.

Proposition 4.9. *Assume that $n \leq 2$. Then R_1 and P_1 are empty.*

Proof. Assume that x is a point in R_1 or P_1. This implies that $\omega_1 \subseteq \mathcal{F}_2$. If $n = 1$, since $b(x) = 0$, one must have $\omega_1 = \omega_2$, hence $\mathcal{F}_1 = \mathcal{F}_2$ and then $hasse_1(x) = 0$. This is a contradiction.
Assume now that $n = 2$. Taking the orthogonal of the inclusion $\omega_1 \subseteq \mathcal{F}_2$ in $\mathcal{E}[\pi]$, one has $\mathcal{F}_1 \subset \omega_2$. As $b = 0$ we have $\omega_1 \subset \omega_2$, thus $\omega_1^{(p)} \subset \omega_2^{(p)}$ and thus $\mathcal{F}_1 \subset \mathcal{F}_2$. In particular ω_1 and $\mathcal{F}_1$ are distincts isotropic lines, and ω_2 is the orthogonal of ω_1, thus one can see that the modified pairing induced on ω_2 is zero, which is not possible. □

Let us now state the principal result on the stratification of the variety. We will need the following remark.

Remark 4.10. Let $S_0 = \operatorname{Spec}(R)$ be a characteristic p scheme, and $T = \operatorname{Spec}(S)$, with $R = S/I$ for some ideal I a thickening of S_0, and assume T is of characteristic p again. Let G be a p-divisible group over S_0 and assume that $I^2 = 0$ in T and denote $\mathcal{E}$ its crystal on the crystalline site $S_0/\operatorname{Spec}(\mathbb{Z}_p)$. Then by Grothendieck-Messing, lifting G to T is the same as lifting is Hodge filtration ω_G to $\mathcal{E}_T$. Assume $\widetilde{\omega}_G \subset \mathcal{E}_T$ is such a lift, then as $I^2 = 0$ we claim that $\widetilde{\omega}_G^{(p)}$ doesn't depend on the lift. Indeed, let $w_1, w_2 \in \mathcal{E}_T$ which both lift $w \in \mathcal{E}_{S_0}$ and let $\underline{e}$ be a basis of $\mathcal{E}_T$ as S-module. Then $w_2 = w_1 + M \cdot \underline{e}$ for some $M \in M_{2h}(I)$. Then $w_2 \otimes 1 = w_1 \otimes 1 + (M\underline{e}) \otimes 1 = w_1 \otimes 1 + \underline{e} \otimes M^\sigma$. But if $i \in I$, and $\sigma = \sigma_T$ is the Frobenius of T (which lifts the one of S_0) then $\sigma(i) = i^p \equiv 0$ in S, thus $w_2 = w_1$. In particular, in the previous situation as both F, V are maps on the crystal $\mathcal{E}$, we see that the lifts of $\mathcal{F}_1, \mathcal{F}_2$ doesn't depend on the lift of ω.

Theorem 4.11. *Assume that $n \geq 2$. The strata X^{ord} and B_0 are open. The strata P_2, B_2 are closed. Moreover*

$$\overline{X^{ord}} = X^{ord} \cup_{i=1}^{2} R_i \cup_{i=0}^{2} P_i \qquad \overline{R_2} = R_2 \cup P_2$$

$$\overline{B_0} = \cup_{i=0}^2 B_i \cup_{i=0}^2 P_i \qquad \overline{B_1} = B_1 \cup \ P_1 \cup P_2 \qquad \overline{P_0} = \cup_{i=0}^2 P_i$$

If $n \geq 3$, one has moreover

$$\overline{R_1} = \cup_{i=1}^2 R_i \cup_{i=1}^2 P_i \qquad \overline{P_1} = P_1 \cup P_2$$

Proof. The fact that X^{ord} and B_0 are open is clear, as is the closeness of P_2. Let us prove the closure relations by looking at where we can specialize (for B_2) or deform points of X.

- If x is a point of B_2, it can only specialize to a point in B_2 or P_2. We need to show that the latter cannot happen. Assume that a point x in $P_2(k)$ can be deformed to $k[[X]]$[2], such that the generization lies in B_2. Since $hasse_1 = 0$, $\omega_1 = \mathcal{F}_1$ over $k[[X]]$. If e_1 is a basis of ω_1, let $u = \{e_1, e_1\}$. The composition of V with the division by π defines a map $V_\pi : \mathcal{E}[\pi] \to \mathcal{E}[\pi]^{(p)}$; similarly, one has a map $F_\pi : \mathcal{E}[\pi]^{(p)} \to \mathcal{E}[\pi]$ given by the composition of the division by π and the Frobenius. These maps are well defined because the image of V is $\mathcal{E}[\pi]$. There exists a unit $u \in O_F^\times$ such that $F_\pi \circ V_\pi = u\,\mathrm{id}$. Since one has $\{F_\pi x, y\} = \{x, V_\pi y\}$, and $V_\pi e_1 = \lambda e_1$ for some unit λ, one finds the equation $u = \lambda_0 u^p$, with $\lambda_0 \in k^\times$. One gets a contradiction, since u must be non zero and divisible by X.

- Let $x \in R_2(k)$. This implies that $\omega_1 = \mathcal{F}_1$, and $\omega_2 = \mathcal{F}_2$. One can find a basis $e_1, \dots, e_{n+1}$ of $\mathcal{E}[\pi]$ such that ω_1 is spanned by e_1, and ω_2 by $e_1, \dots, e_n$, and the modified pairing is given by the matrix
$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & J_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \text{with } J_{n-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1}$$
One then look for a lift to $k[[T]]$, first of the Hodge filtration together with the extra data. The line ω_1 can be lifted to a line spanned by a vector $\begin{pmatrix} 1 \\ X \\ y \end{pmatrix}$. The vector needs to be isotropic, hence the condition
$$2y +^t XJ_{n-1}X = 0$$
Then we will look at the corresponding deformation step by step, i.e. successively from $k[T]/(T^n) \longrightarrow k[T]/(T^{n-1})$ which is given by a square zero ideal. At each step, we have a p-divisible group G_n over $k[T]/(T^n)$ and by remark 4.10, the deformation to G_{n+1} has a canonical lift of $\mathcal{F}_1$ and $\mathcal{F}_2$ which we can assume, if $\mathrm{hasse}_1(G_n) = \mathrm{hasse}_2(G_n) = 0$ given by e_1 and $e_1, \dots, e_n$. The condition for the generization to be in R_2 is that at each step $X = 0$, $y = 0$. The condition for it to be in R_1 is $y = 0$. Since $n \geq 2$, the point can always be lifted to a point in X^{ord}. If $n \geq 3$, it can be lifted to a point in R_1, but as in this case the two conditions (ω_1 totally isotropic and $\omega_1 \subset \mathcal{F}_2$) make a non smooth condition, let us give a more precise argument : set $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_k k[[t]]$ and choose a lift of the basis such that the pairing is of the previous form, and reducing on $k[t]/(t^2)$ we have $\mathcal{F}_1$ given by e_1 and $\mathcal{F}_2$ by $e_1, \dots, e_n$, as

[2] In particular this means that we have a deformation of the Hodge filtration, and conversly a deformation of the Hodge filtration to $k[[X]]$ induces step by step by Grothendieck-Messing a deformation of the p-divisible group.

before. Then set $\widetilde{\omega}_1$ spanned by $\begin{pmatrix} 1 \\ t \\ 0_{n-1} \end{pmatrix}$. Then clearly $\widetilde{\omega}_1$ is totally isotropic, and reducing modulo t^2 we see that $\widetilde{\omega}_1 \neq \mathcal{F}_1 \mod t^2$, thus our deformed point is not in R_2 anymore, but we can't assure that the $k[[t]]$-point is in R_1 at the moment. So assume we have lifted ω_1 to $k[t]/(t^n)$ to a point in R_1, and we moreover assume that there is a basis of $\mathcal{E} \otimes_k k[t]/(t^n)$ such that ω_1 is spanned by $\begin{pmatrix} 1 \\ t \\ 0_{n-1} \end{pmatrix}$. We then choose a lift of this basis to $\mathcal{E} \otimes_k k[t]/(t^{n+1})$ such that the pairing as the same form (1). Then we simply set again $\widetilde{\omega}_1$ to be spanned by $\begin{pmatrix} 1 \\ t \\ 0_{n-1} \end{pmatrix}$. Inducting the argument gives the resulting point in R_1.

- The fact that any point $x \in P_0(k)$ can be deformed to a point in X^{ord} or B_0 follows from the previous sections.

- A point $x \in P_2(k)$ can be deformed to P_0 (and hence B_0, X^{ord}), and P_1 if $n \geq 3$, with exactly the same arguments as before, as we never used that $m \neq 0$. If we want to deform x to R_2, we can lift ω_1 "trivially" so that $\omega_1 \subset \omega_2 := \omega_1^{\perp'}$, and then deform ω/ω_2 so that $\pi\omega \subset \omega_1$ (by choosing elements as in Proposition 2.11). We can then deform to R_1 if $n \geq 3$. The point x can also be deformed to B_1, by lifting ω_1 inside $\mathcal{F}_2$, non isotropically : concretely choose the isotrivial lift to $k[[T]]$ of $\widetilde{\omega}$ of ω (here it means it is still of p-torsion, i.e. $\widetilde{\omega} = \widetilde{\mathcal{E}}[\pi]$), and then inductively for each n, there is a canonical lift of $\mathcal{F}_1, \mathcal{F}_2$ from $k[T]/(T^{n-1})$ to $k[T]/(T^n)$ by remark 4.10 if ω_1 and thus $\omega_2 = \omega_1^{\perp'}$ have been deformed to $k[T]/(T^{n-1})$. We thus have deformations of $\mathcal{F}_1, \mathcal{F}_2$ to $k[T]/(T^n)$ (orthogonal to each other for the modified pairing) and we choose a deformation of ω_1 still assuming $\widetilde{\omega}_1 \subset \mathcal{F}_2$. This is possible as the Grassmanian $\mathrm{Gr}_a(\mathcal{F}_2)$ is smooth. If n is big enough, as the condition of being totally isotropic is a closed condition which defines a proper closed subspace of $\mathrm{Gr}_a(\mathcal{F}_2)$, there exists a deformation of ω_1 which is not isotropic anymore. After this choice, any lift of ω_1 will do. The corresponding deformed p-divisible group is in B_1. Note that we have already proven that we can't deform from P_2 to B_2.

- A similar but easier argument shows that we can deform from B_2 to B_1, and it is easy to see that any element $x \in B_1(k)$ can be deformed to B_0.

- To finish the proof, let us remark that a point in R_1 can be deformed to X^{ord} if $n \geq 3$, by lifting ω_1 isotropically outside $\mathcal{F}_2$. Indeed, we have $\pi\omega = \omega_1 \subset \omega_2$ and by hypothesis $\mathcal{F}_1 \neq \omega_1 \subset \mathcal{F}_2$. In particular $\omega_2 \neq \mathcal{F}_2$. The divided pairing, on a basis $e_1, \dots, e_h$ such that e_1 generates ω_1 and $e_1, \dots, e_{h-1}$ generates ω_2 can be given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We thus look for a lift of ω_1 given by a vector $\begin{pmatrix} 1 \\ X \\ y \end{pmatrix}$ with y, X with coefficients in $tk[[t]]$. This lift is totally isotropic if $2y + {}^tXX = 0$. Let us prove that we can choose it away from

$\mathcal{F}_2$. As mod t, $\omega_1 \subset \mathcal{F}_2 \neq \omega_2$, we have $e_1 \in \mathcal{F}_2$, there exist $e_i \notin \mathcal{F}_2$ and as $n \geq 2$, there is a non zero vector of the form

$$v = \begin{pmatrix} 0 \\ B \\ 0 \end{pmatrix} \in \mathcal{F}_2.$$

Thus if we set $\widetilde{\omega}_1$ generated by

$$v = \begin{pmatrix} 1 \\ tB + ta\delta_i \\ 0 \end{pmatrix} \notin \mathcal{F}_2.$$

for a non zero a, the condition of being totally isotropic is given by $t^2(\sum_j b_j^2 + 2b_i a + a^2) = 0$. In particular if $\sum_j b_j^2$ is non zero or if $b_i \neq 0$ we can find such a non zero a. So assume that $\sum_j b_j^2 = 0$ but $b_i = 0$. As v is non zero, there is j such that $b_j \neq 0$. If $e_j \notin \mathcal{F}_2$ then the previous argument applies. Otherwise $e_j \in \mathcal{F}_2$, and thus $w = v + ce_j \in \mathcal{F}_2$. But if we calculate its norm for the divided pairing, we have $\sum_i b_i^2 + 2cb_j + c^2 = 2cb_j + c^2$. But we can find c such that this is non zero, and then reapply the previous argument with w instead of v.

- Finally, if $n \geq 3$, one checks that a point in P_1 can be deformed to P_0 (and then X^{ord}, B_0) by the exact same calculation. We can also deform from P_1 to B_1 : mod t we have $\omega_1 \subset \omega_2 \neq \mathcal{F}_2$, thus up to choosing a basis as before we can set $\widetilde{w}_1$ mod t^2 generated by

$$\begin{pmatrix} 1 \\ tB \\ t \end{pmatrix} \in \widetilde{\mathcal{F}_2},$$

where

$$v = \begin{pmatrix} 0 \\ B \\ 1 \end{pmatrix} \in \mathcal{F}_2.$$

Then clearly $\widetilde{\omega}_1$ is not isotropic. Then assume that we have lifted ω_1 to $k[t]/(t^n)$, this gives a lift of $\mathcal{F}_2$ to $k[t]/(t^{n+1})$ and we choose any lift of ω_1 inside this. By induction, and Grothendieck-Messing, we get a point in B_1. We can also deform from P_1 to R_1 : assume that we have lifted ω_1 to $k[t]/(t^n)$, inside $\mathcal{F}_2$, which has a canonical lift mod t^{n+1}. Then we want to deform ω_1 isotropically while staying in $\mathcal{F}_2$. But as $\omega_1^\perp \cap \mathcal{F}_2$ is non trivial in special fiber, we can indeed find a lift of $\omega_1 \subset \mathcal{F}_2$ at each step which remains isotropic.

□

4.4 Stratification when $n = 1$

We now suppose that $n = 1$. In this case P_1 and R_1 are empty. The situation is the following.

Theorem 4.12. *The strata X^{ord}, R_2 and B_0 are open. The strata P_0, P_2 and B_i $(i = 1, 2)$ are closed. Moreover*

$$\overline{X^{ord}} = X^{ord} \cup P_0 \qquad \overline{R_2} = R_2 \cup P_2 \qquad \overline{B_0} = \cup_{i=0}^2 B_i \cup P_0 \cup P_2$$

Proof. It is clear that X^{ord} and B_0 are open. As previously, any point of P_0 can be deformed to X^{ord} or B_0. It is easy to see that any point of B_i $(i = 1, 2)$ can be deformed to B_0.
Let us prove that R_2 is open. Let $x \in R_2(k)$, and let us investigate the possible lifts of x to a ring R. Over this ring, the space $\mathcal{F}_1$ lifts canonically. By assumption, the space ω_1 is equal to its $\mathcal{F}_1$ over k. Since any lift of ω_1 must be isotropic, and $\mathcal{E}[\pi]$ is a 2-dimensional space with a perfect pairing, we see that the space of totally isotropic lines in it is zero dimensional and reduced, thus one must have an equality $\widetilde{\omega}_1 = \mathcal{F}_1$. It is then not possible to lift x to a point in X^{ord}.
The same arguments show that a point in P_2 cannot be deformed into P_0 or X^{ord}. Similarly, if $x \in P_2$ is deformed over $k[[t]]$ in B_1 or B_2, for each n, modulo t^n this implies that we have canonical lifts $\mathcal{F}_1, \mathcal{F}_2$ modulo T^{n+1}. If $\omega_1 = \omega_2 \mod t^n$, then $\mathcal{F}_1 = \mathcal{F}_2$, and if we deform in B_1 or B_2 (or any point such that $\text{hasse}_2 = 0$) we must have $\widetilde{\omega}_1 \subset \mathcal{F}_2$, but they have the same rank thus an equality, and thus $\widetilde{\omega}_1 = \mathcal{F}_1 = \mathcal{F}_2 = \widetilde{\omega}_2$. Thus actually the deformation remain in P_2. This proves that points of P_2 can only be possibly deformed to a point in R_2 or B_0. Conversely we can indeed deform to R_2 by only deforming ω/ω_2 to make it non-π-torsion as in proof of proposition 2.11. To deform a point of P_2 to B_0, it is enough to deform $\omega_1 \subset \omega = \mathcal{E}[\pi]$ by a non-totally isotropic line. This is possible as this space is smooth (it is a projective space of dimension > 0). □

5 Case of a general CM field F

Let $(B, \star, V, <,>, h)$ be P.E.L. datum (see [Lan13]), so that $B/\mathbb{Q}$ be a finite dimensional central semi-simple $\mathbb{Q}$-algebra, with involution $\star$, center F.

Example 5.1. Let F_0 be a totally real field, and F/F_0 a CM field. Take $B = F$, $\star = c$ the complex conjugacy, $V = F^n$ and polarisation by $(x, y) = xJc(y)$ for an hermitian matrix J. Let p be a prime. Then $B_{\mathbb{Q}_p} = \prod_{\pi_0|p \in F} F \otimes_{F_0} F_{0,\pi}$. Everything splits over primes above p in F_0, thus for simplicity, we can assume that there is only one prime π_0 of F_0 above p. Let e, f be the ramification index, and the residual degree of π_0 over p. The case of unramified primes in F/F_0 is treated in [BH22], thus we can assume that π_0 ramifies in $F_p := F \otimes \mathbb{Q}_p$ and choose it so that $\pi_0 = \pi^2$ for some uniformizer π of F_p.

Now fix an integral P.E.L. datum $(\mathcal{O}_B, \star, \Lambda, <,>)$, so that in particular $\mathcal{O}_B$ is a $\mathbb{Z}_{(p)}$-order in B, $\star$-stable and maximal over $\mathbb{Z}_p$, and $(\Lambda, <,>) \otimes_{\mathbb{Z}} \mathbb{Q} = (V, <,>)$.

Hypothesis 5.2. We assume the following

1. $B_{\mathbb{Q}_p}$ is a product of matrix algebra over finite extension of $\mathbb{Q}_p$.
2. p is a *good* prime, i.e. $p \nmid [\Lambda^\sharp, \Lambda]$.

To simplify we assume that $\star$ is of the second kind on each simple factors of $(B, \star)$ (in particular we exclude factors of type D see [BH22], Hypothesis 2.2) : factors of type (C) can be dealt with as in [BH22]. In most of what follows, we can treat simple factors separately, so that we will be able to assume $(B_{\mathbb{Q}_p}, \star)$ is a matrix algebra over its center or a product of two isomorphic matrix algebras over a field exchanged by $\star$. This second case is treated in [BH22]. So to fix notations we will often assume that $B_{\mathbb{Q}_p} = M_n(F_\pi)$, for some finite extension F_π of $\mathbb{Q}_p$, and we denote $F_{\pi_0} = F_p^{\star=1}$: the extension F_π/F_{π_0} is of degree 2. Moreover, we assume that the local field extension F_π/F_{π_0} is ramified (otherwise this is treated in [BH22] again). Fix an uniformizer π_0 of F_{π_0} and π of F so that $\pi^2 = \pi_0$. Let $F_{\pi_0}^{ur}$ be the maximal unramified extension contained in F_{π_0}, and $\mathcal{T}$ the set of embeddings of $F_{\pi_0}^{ur}$ into $\overline{\mathbb{Q}_p}$. For each $\tau \in \mathcal{T}$, let Σ_τ be the set of embeddings of F_{π_0} extending τ.

We write $\Sigma_\tau = \{\sigma_{\tau,1}, \dots, \sigma_{\tau,e}\}$. For each $\tau \in \mathcal{T}$ and $1 \leq i \leq e$, let $\sigma_{\tau,i}^+$ and $\sigma_{\tau,i}^-$ be the embeddings of F_π extending $\sigma_{\tau,i}$: these are notations which will remain in force everytime we (implicitely) choose a simple factor of $B_{\mathbb{Q}_p}$.

As in [BH22], Section 2.2, we can associate to the Shimura data a combinatorial data $(d_{j,\tau'})_{\tau'}$, (j corresponding to a choice of a simple factor) which, when we restrict to a simple factor of the previous type, is just a collection $(d_\sigma)_{\sigma: F_\pi \hookrightarrow \overline{\mathbb{Q}_p}}$ satisfying $d_{\sigma \circ c} = h - d_\sigma$ for a fixed value of $h \geq 1$ (which might depend on the simple factor). For simplicity we denote for $\tau \in \mathcal{T}, i = 1, \dots, e$, $a_{\tau,i} = d_{\sigma_{\tau,i}^+}, b_{\tau,i} := d_{\sigma_{\tau,i}^-}$.

Definition 5.3. Let Y be the moduli space over O_F whose R-points are equivalence classes of tuples $(A, \lambda, \iota, \eta, \omega_1)$ up to $\mathbb{Z}_{(p)}^\times$-isogenies, where

- A is an abelian scheme over R
- λ is a $\mathbb{Z}_{(p)}^\times$-polarization
- $\iota : O_B \to End(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, making the Rosati involution and $\star$ compatible
- η is a rational Λ-level structure outside p
- For every simple factor $j = M_n(F_\pi)$ of $B_{\mathbb{Q}_p}$, there is an associated direct factor ω_j' of ω_A. By Morita equivalence, we have a $\mathcal{O}_{F_\pi}$-module $\omega_j = \bigoplus_\tau \omega_{\tau,j}$. We then ask for
$$0 = \omega_\tau^{[0]} \subseteq \omega_\tau^{[1]} \subseteq \cdots \subseteq \omega_\tau^{[e]} = \omega_{\tau,j},$$
is a PR filtration, meaning that each $\omega_\tau^{[i]}$ is locally a direct factor, stable by O_{F_π}.
- the quotient $\omega_\tau^{[i]}/\omega_\tau^{[i-1]}$ is locally free of rank h.
- $O_{F_{\pi_0}}$ acts by $\sigma_{\tau,i}$ on $\omega_\tau^{[i]}/\omega_\tau^{[i-1]}$.
- the filtration is compatible with the polarization
- For each i, $\omega_\tau^{[i-1]} \subseteq \omega_{\tau,1}^{[i]} \subseteq \omega_\tau^{[i]}$, where $\omega_{\tau,1}^{[i]}$ is locally a direct factor stable by O_{F_π}.
- $\omega_{\tau,1}^{[i]}/\omega_\tau^{[i]}$ is locally free of rank $a_{\tau,i}$, and O_{F_π} acts by $\sigma_{\tau,i}^+$ on it, and by $\sigma_{\tau,i}^-$ on the quotient $\omega_\tau^{[i+1]}/\omega_{\tau,1}^{[i]}$ (which is automatically locally free of rank $b_{\tau,i}$).

Let us be more precise about the compatibility with the polarization. One has a pairing on $\mathcal{E}$, and $\omega_{\tau,j}$ is totally isotropic for this pairing. The compatibility for the filtration is that

$$(\omega_\tau^{[i]})^\perp = Q_\tau^i(\pi_0)^{-1}\omega_\tau^{[i]}, \quad Q_\tau^i(T) = \prod_{t=i+1}^{e} (T - \sigma_{\tau,t}(\pi_0)),$$

and $Q_\tau = \prod_{i=1}^e (T - \sigma_{\tau,i}(\pi_0))$ is a minimal polynomial for π_0 in $\tau(F_{\pi_0}^{ur})$. Let us define $\mathcal{E}_\tau^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1}\omega_\tau^{[i-1]}/\omega_\tau^{[i-1]}$. It is a locally free sheaf with an action of O_F, and an alternating perfect pairing. One has the subsheaves $\mathcal{F}_\tau^{[i]} := \omega_\tau^{[i]}/\omega_\tau^{[i-1]}$, which is totally isotropic for the previous pairing, and $\mathcal{F}_{\tau,1}^{[i]} := \omega_{\tau,1}^{[i]}/\omega_\tau^{[i-1]}$. We define $\mathcal{F}_{\tau,2}^{[i]} := (\pi - \sigma_{i,\tau}^-(\pi))(\mathcal{F}_{\tau,1}^{[i]})^\perp$.

Definition 5.4. Let k be an algebraically closed field of characteristics p. Let $x \in Y(k)$, and let τ, i. We define the integers $h_\tau^{[i]}$ and $l_\tau^{[i]}$ as the dimensions of $\pi\mathcal{F}_\tau^{[i]}$ and $\mathcal{F}_{\tau,1}^{[i]} \cap \mathcal{F}_{\tau,2}^{[i]}$ respectively.

Let $C := \{(h_\tau^{[i]}, l_\tau^{[i]})_{\tau\in\mathcal{T},1\leq i\leq e}, 0 \leq h_\tau^{[i]} \leq l_\tau^{[i]} \leq \min(a_{\tau,i}, b_{\tau,i})\}$. We define a stratification on $X = Y \times \operatorname{Spec}(k_F)$ by

$$X = \coprod_{c\in C} X_c,$$

where $X_c = \{x \in X(k) | (h_\tau^{[i]}(x), l_\tau^{[i]}(x)) = c\}$. Let $c = (h_\tau^{[i]}, l_\tau^{[i]})$ and $c' = (h_\tau^{[i]\prime}, l_\tau^{[i]\prime})$ be elements of C. We say that $c \leq c'$ if for all τ, i,

$$h_\tau^{[i]\prime} \leq h_\tau^{[i]} \leq l_\tau^{[i]} \leq l_\tau^{[i]\prime}$$

Theorem 5.5. *One has*

$$\overline{X_c} = \coprod_{c'\leq c} X_{c'}$$

Proof. When deforming a point of Y, one has to deform the Hodge filtration. We do it one τ at a time as follows.
By the results of the previous section, we can deform both $\omega_{\tau,1}^{[1]} \subseteq \omega_\tau^{[1]}$ inside $\widetilde{\mathcal{E}}_\tau^{[1]} := \mathcal{E}_\tau^{[1]} \otimes_k k[[t]]$, with the deformation $\widetilde{\omega}_\tau^{[1]}$ of $\omega_\tau^{[1]}$ isotropic (for the divided pairing) and with $(h_\tau^{[1]}, l_\tau^{[1]}) = (h_\tau^{[1]\prime}, l_\tau^{[1]\prime})$. This is the result of Proposition 2.11. Then, look at $\pi^{2(e-1)}\omega_\tau^{[1]}/\omega_\tau^{[1]}$: this space has a natural lift $\pi^{2(e-1)}\widetilde{\omega}_\tau^{[1]}/\widetilde{\omega}_\tau^{[1]}$ inside $\widetilde{\mathcal{E}}/\widetilde{\omega}_\tau^{[1]}$. We then take a isotrivial lift of the filtration $\dots \subset \omega_\tau^{[i-1]}/\omega_\tau^{[1]} \subset \omega_{\tau,1}^{[i]}/\omega_\tau^{[1]} \subset \omega_\tau^{[i]}/\omega_\tau^{[1]} \subset \dots$, for $e \geq i \geq 1$ and then pull back to $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_k k((t))$ to get a full lift, and we get a point over $k((t))^{perf}$ with new $\widetilde{c}_\tau^{[1]} = c_\tau^{[1]\prime}$ but $\widetilde{c}_\tau^{[i]} = c_\tau^{[i]}$ for $i \geq 2$. Then by induction, we can assume that for $1 \leq s \leq i$ we have $c_\tau^{[s]} := (h_\tau^{[s]}, l_\tau^{[s]}) = (h_\tau^{[s]\prime}, l_\tau^{[s]\prime}) =: c_\tau^{[s]\prime}$ for our point over k. Then one deforms $\omega_{\tau,1}^{[i+1]}/\omega_{\tau,1}^{[i]} \subset \omega_\tau^{[i+1]}/\omega_{\tau,1}^{[i]}$ inside $\widetilde{\mathcal{E}}_\tau^{[i]}$ again using Proposition 2.11, and we do the same isotrivial lift for the rest of the filtration as when $i = 0$, to get the induction step, and thus the result. □

Theorem 5.6. *Y is normal and flat over $\mathcal{O}_F$, and its special fiber is a reduced, local complete intersection.*

Proof. Consider again a local model diagram as in the proof of Proposition 2.15. The local model splits over direct factors of $B_{\mathbb{Q}_p}$, thus it is sufficient to show the theorem for one such factor only. Let $F_\pi/F_{\pi_0}, e, f$ etc. as before, $M_n(\mathcal{O}_{F_{\pi_0}})$ the factor, and denote Λ' the part of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p$ corresponding to this simple factor and using Morita equivalence (so that $\Lambda = \sum_j \mathcal{O}_{F_{\pi_0}}^n \otimes_{\mathcal{O}_{F_{\pi_0}}} \Lambda'$). We have a diagram

$$Y \longleftarrow \widetilde{Y} = \operatorname{Isom}(\mathcal{E}, \Lambda \otimes \mathcal{O}_S) \longrightarrow \mathcal{N},$$

where the first map is a torsor over a smooth group scheme $\mathcal{G}$, and the second map is formally smooth and $\mathcal{G}$-equivariant by Grothendieck-Messing. Here $\mathcal{N}$ is a local model, see e.g. [PR05] section 14, analogous to the one in the proof of Proposition 2.15, parametrizing

- A PR-filtration $0 = F_\tau^{[0]} \subset F_\tau^{[1]} \subset \dots \subset F_\tau^{[e]} = \Lambda_{\tau,j} \otimes \mathcal{O}_S$ in $\Lambda \otimes \mathcal{O}_S$, each $F_\tau^{[i]}$ is a locally direct factor, stable by $\mathcal{O}_{F_\pi}$.
- Each quotient $F_\tau^{[i]}/F_\tau^{[i-1]}$ is locally free of rank $h = a_{\tau,i} + b_{\tau,i}$ and $\mathcal{O}_{F_{\pi_0}}$ acts by $\sigma_{\tau,i}$ on it.

- The filtration is compatible with the polarisation
- For each i, a locally direct factor $F^{[i-1]} \subset F_1^{[i]} \subset F^{[i]}$, stable by $\mathcal{O}_{F_\pi}$,
- $F_{\tau,1}^{[i]}/F_\tau^{[i-1]}$ is a locally direct factor of rank $a_{\tau,i}$ and $\mathcal{O}_{F_\pi}$ acts through $\sigma_{\tau,i}^+$
- $\mathcal{O}_{F_\pi}$ acts through $\sigma_{\tau,i}^-$ on $F^{[i]}/F_{\tau,1}^{[i]}$ (and this is automatically locally free of rank $b_{\tau,i}$).

$F_\tau^{[i]}$ is obviously the analog of the $\omega_\tau^{[i]}$ in the definition of Y, and $F_{\tau,1}^{[i]}$ of $\omega_{\tau,1}^{[i]}$. Thus it is enough to see that $\mathcal{N}$ is flat, normal, and its special fiber is a local complete intersection. The theorem 5.5 actually shows that

$$N := \overline{\mathcal{N}} = \coprod_c N_c,$$

with expected (strong) closure relations. The proof of Proposition 2.12 for the maximal strata carries over and shows (doing one $F_{\tau,1}^{[i]}$ at a time) that maximal strata of N are smooth, thus reduced, and $\mathcal{N}$ is smooth in codimension 1. For each i, we have a space $\mathcal{N}_{\leq i}$ parametrizing locally direct factors $F_\tau^{[1]} \subset \dots \subset F_\tau^{[i]} \subset \Lambda_{\tau,j} \otimes \mathcal{O}_S$ with the same properties as before, together with $\mathcal{F}_{\tau,1}^{[k]}$ in $F_\tau^{[k]}/F_\tau^{[k-1]}$ of rank $a_{\tau,k}$ such that the actions of $\mathcal{O}_{F_\pi}$ is through $\sigma_{\tau,k}^+$ on $\mathcal{F}_{\tau,1}^{[k]}$ and by $\sigma_{\tau,k}^-$ on the cokernel of the inclusion, for $k = 1, \dots, i$. We have a natural maps

$$\mathcal{N} = \mathcal{N}_{\leq e} \longrightarrow \mathcal{N}_{\leq e-1} \longrightarrow \dots \longrightarrow \mathcal{N}_{\leq 1} \longrightarrow \mathrm{Spec}(\mathcal{O}_F) =: \mathcal{N}_{\leq 0}.$$

We will show that each of these maps is a relative LCI in special fiber. This will then show that $\mathcal{N}$ is LCI in special fiber, thus everywhere, and as $\mathcal{N}$ is smooth in codimension 1, that $\mathcal{N}$ is normal, and that the map to $\mathcal{O}_F$ is flat (by miracle flatness).

As $\mathcal{N}$ and $\mathcal{N}_{\leq i}$ decomposes naturally as product over the simple factors of $B_{\mathbb{Q}_p}$, and over the index τ, thus we can assume that there is only one factor and that $\mathcal{O}_B \otimes \mathbb{Z}_p = \mathcal{O}_F$ and that $\mathcal{T} = \{\tau\}$ so we suppress τ from the notations. Denote $E^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1}F^{[i-1]}/F^{[i-1]}$, endowed with its own (perfect) pairing. By definition $\mathcal{N}_{\leq i}$ over $\mathcal{N}_{\leq i-1}$ parametrizes locally direct factors $F_1^{[i]}$ and $F^{[i]}$ of $E^{[i]}$ of respective ranks a_i and $h = a_i + b_i$, such that moreover $F_1^{[i]} \subset F^{[i]}$ and $F_1^{[i]} \subset E^{[i]}[\pi - \sigma_i^+(\pi)]$. So let $U \subset \mathcal{N}_{\leq i-1}$ for $i \geq 1$ a small affine so that all $F^{[k]}, k < i$ and $E^{[i]} := (\pi_0 - \sigma_{\tau,i}(\pi_0))^{-1}F^{[i-1]}/F^{[i-1]}$ are free. But over U, $\mathcal{N}_{\leq i} \times_{\mathcal{N}_{\leq i-1}} U$ is locally isomorphic in special fiber to the special fiber of a space $\mathcal{N}' \otimes_{\mathcal{O}_F} U$, with $\mathcal{N}'$ as in the proof of proposition 2.15. Indeed, we already have that $E^{[i]}$ is a locally free $\mathcal{O}_{F_\pi}/(\pi_0) \otimes \mathcal{O}_S$-module, but

$$\mathcal{O}_{F_\pi}/(\pi^0) = \mathcal{O}_{F_{\pi_0}}[X]/(X^2 - \pi_0)/(\pi_0) = \mathcal{O}_{F_{\pi_0}}[X]/(X^2, \pi_0) = \mathcal{O}_{F^{nr}}[X]/(X^2, p),$$

where $F^{nr} \subset F_{\pi_0}$ is the maximal unramified extension. Thus if we choose K a degree 2 ramified extension of F^{nr}, we have a module over $\mathcal{O}_K/p \simeq \mathcal{O}_{F^{nr}}[X]/(X^2, p)$. Now we claim that we can make the pairing of $E^{[i]}$ locally trivial. First, at it is perfect, we choose a basis so that it is of the form

$$\begin{pmatrix} 0 & \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_h \end{pmatrix} \\ \begin{pmatrix} -a_1 & & \\ & \ddots & \\ & & -a_h \end{pmatrix} & 0 \end{pmatrix},$$

with $a_i \in \mathcal{O}_S^\times$. But now, up to changing the basis vectors $(e_1, \dots, e_h, f_1, \dots, f_h)$ by $(e_1, \dots, e_h, a_1^{-1} f_1, \dots, e_h^{-1} f_h)$ it is of the desired form. We are thus reduced to the case of $\mathcal{N}' \times_{\mathcal{O}_F} U$ with K (a degree 2 totally ramified extension) instead of F, whose special fiber is a relative complete intersection over U by Proposition 2.15. Thus $\mathcal{N}$ is a LCI in special fiber, thus it is LCI. □

6 The case when $p = 2$.

In this section, we will investigate the case $p = 2$.

6.1 Quadratic forms in characteristic 2

Let k be an algebraically closed field of characteristic 2, let V be a k-vector space of dimension d. Let $<,>$ be a non-degenerate symmetric bilinear form, and let q be the associated quadratic form defined by $q(x) =< x, x >$.

Proposition 6.1. *Up to isomorphism, we are in one of the two following situations:*

1. *q is not identically zero, and the matrix of the bilinear form is the identity matrix in a certain basis.*
2. *q is identically zero. This implies that d is even, and the matrix of the bilinear form in a certain basis is of the form*

$$\begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Proof. One proves this result by induction on the dimension. If the dimension is 1 or 2, it is an easy computation.
Assume the result true for all $k \leq d-1$, and let us prove it for d. Assume that q is identically zero. Take a vector e_1 in V and a vector e_2 such that $< e_1, e_2 >= 1$. Let F be the orthogonal of the vector space spanned by e_1, e_2. Applying the induction hypothesis to F gives the result.
Assume now that q is not identically 0. Let e_1 be a vector, normalized such that $< e_1, e_1 >= 1$. Let F be the orthogonal of the space generated by e_1. One can apply the induction result to F. This gives the result, noticing that the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are equivalent. Indeed, if e_1, e_2, e_3 is a basis for which the matrix is the second one, then the change of basis $e_1' = e_1 + e_2 + e_3$, $e_2' = e_1 + e_2$, $e_3' = e_2 + e_3$ gives the first matrix. □

6.2 Geometry in the first case

In this section we assume that we are in the first case, i.e. the modified pairing given on $A[\pi]$ is given by the identity matrix.

Proposition 6.2. *The smooth locus is $X_{0,0}$, which has dimension ab. Moreover, the other strata $X_{(h,\ell)}$, $\ell \neq 0$, are non-smooth.*

Proof. It is clear that the points in $X_{0,0}$ are smooth as this is open in X, and square-zero deformations corresponds to deforming in a Grassmanian $\mathcal{G}r_{a,a+b}$, this gives also the dimension. To prove that these are the only smooth points, we'll argue as the case of characteristics $p \neq 2$, but first we need a lemma.

Claim 6.3. Let $V, (,)$ be a non-degenerate symmetric space of rank N over k algebraically closed of characteristics 2 with quadratic form q non-zero. Let $W \subset V$ totally isotropic of rank h. Then we claim that there exists a basis $\underline{e}$ of V such that the matrix of $(,)$ is I_n and $W = \mathrm{Vect}(e_1 + e_2, e_3 + e_4, \dots, e_{2h-1} + e_{2h})$.

Proof of claim. Indeed, let $W = \mathrm{Vect}(f_1, \dots, f_h)$ and $g_1, \dots, g_h$ such that $(g_i, f_j) = \delta_{i,j}$ (this is possible by pulling back the dual basis of a completion of $\underline{f}$). We claim that we can modify the g_i so that $(g_i, g_j) = \delta_{i,j}$. If (g_1, g_1) is non zero then we can rescale to get $(g_1, g_1) = 1$. Otherwise, there exists $v \in (f_1, \dots, f_h)^\perp$ such that $(v, v) \neq 0$: indeed, if $(v, v) = 0$ then $(f_1, \dots, f_h)$ is totally isotropic, thus there can exists at most $N/2-h$ such vectors, but $\dim(f_1, \dots, f_h)^\perp = N-h$. Setting $g_1 + v$ instead of g_1 we do not change the values on the f_i's but $(g_1 + v, g_1 + v) = (v, v)$ as we are in characteristics 2. Assume that we have constructed $g_1, \dots, g_i, i < h$ such that $(g_k, f_i) = \delta_{k,i}$ for all $k, i \leq h$ and $(g_k, g_\ell) = \delta_{k,l}$ for all $k, \ell \leq i$. We clam that we can find $v \in T = (f_1, \dots, f_h, g_1, \dots, g_i)^\perp$ such that $(v, v) \neq 0$. This space is $N - h - i$-dimensional. If a basis $v_1, \dots, v_{N-h-i}$ of vectors for this space are of norm 0; then they are also orthogonal since $2 = 0$. In particular, $(v_1, \dots, v_{N-h-i})^\perp$ contains $(f_1, \dots, f_h, g_1, \dots, g_i, v_1, \dots, v_{N-h-i})$. This last space has dimension N, thus $T = 0$, which is absurd since $i < h \leq N/2$. Choose a v of non zero norm, then set

$$g'_{i+1} = g_i + \sum_{k=1}^{i} \lambda_k f_k + v.$$

The norm of g'_{i+1} is then $(g_{i+1}, g_{i+1}) + (v, v)$ and, for $k \leq i$,

$$(g_{i+1}, f_k) = 0, \quad (g_{i+1}, g_k) = (g_{i+1}, g_k) + \lambda_k.$$

Thus if we set $\lambda_k = -(g_{i+1}, g_k)$ we have that $(g'_{i+1}, g_k) = 0$ and, up to change v by $\mu v, \mu \in k$ and rescaling g_{i+1}, we can assume that $(g_{i+1}, g_{i+1}) = 1$. Setting $e_{2i-1} = f_i - g_i$ and $e_{2i} = g_i$, we have a beginning of a basis such that $(e_i, e_j) = \delta_{i,j}$. Then, if $h < N/2$, we can look at $\mathrm{Vect}(e_1, \dots, e_{2h})^\perp$ and argue as the end of proof of proposition 6.1 □

Now let $x \in X_{h,l}(k)$ with $\ell \neq 0$. Let us prove that the strata are not smooth at any closed point. Then a $k[t]/(t^n)$ lift of x in $X_{h,l}$ induces a lift of $W = \omega_1 \cap \omega_2 \subset \mathcal{E}[\pi]$, which is totally isotropic. We will find a lift of x mod t^2 which we cannot lift. First, by the claim we can assume that the matrix of the divided pairing is the identity, and W is given in the basis $(e_1, e_3, \dots, e_{2h-1}, e_2, e_4, \dots, e_{2h})$ by

$$\begin{pmatrix} I_h \\ I_h \end{pmatrix}.$$

Now we will choose the deformation such that the lift of W is given by

$$\begin{pmatrix} I_h & \\ & I_h \end{pmatrix} + t \begin{pmatrix} N_1 & \\ & N_2 \end{pmatrix},$$

and we will actually set $N_1 = E_{1,1} + tN_1'$, and $N_2 = tN_2'$. Then the lift of W is totally isotropic if

$$t(N_1 +^t N_1) + t^{2t} N_1 N_1 + t(N_2 + {}^t N_2) + t^{2t} N_2 N_2 = 0,$$

thus the lift of W mod t^2 is indeed totally isotropic, and we check there is indeed a lift to $k[t]/(t^2)$ which gives this lift of W[3]. Now we can show that there is no choice of N_1', N_2' such that this lift to $k[t]/(t^3)$. Indeed, otherwise we would get as equation mod t^3

$$t^2 E_{1,1} + t^2(N_1' + {}^t N_1' + N_2' + {}^t N_2') = 0,$$

but we can check easily that the right matrix always has a zero coefficient in position $(1,1)$ as we are in characteristics 2. This imply that all strata $X_{h,\ell}, \ell \neq 0$ aren't smooth at any point. In particular, as they are open, the strata $X_{h,h}$ are not is the smooth locus. Let us prove the following claim

Claim 6.4. Any point $x \in X_{h,\ell}(k)$ with $\ell \neq 0$ can deformed in $X_{h',\ell}(k)$ with $h' \neq 0$.

Proof of claim. If $h \neq 0$ this is trivial thus assume $h = 0$. We will construct a $k[[t]]$-deformation of x whose generic fiber lies in $X_{1,\ell}$. Choose any basis $\pi e_1, \dots, \pi e_n$ of $\omega = \mathcal{E}[\pi]$ such that $\pi e_1, \dots, \pi e_\ell$ is a basis of $\omega_1 \cap \omega_2$, $\pi e_1, \dots, \pi e_a$ of ω_1 and $\pi e_1, \pi e_\ell, \pi e_{a+1}, \dots, \pi e_{a+b-\ell}$ of ω_2. As $\ell > 0$ we have $\omega_1 + \omega_2 \subsetneq \omega = \mathcal{E}[\pi]$, thus we can assume $\pi e_n \notin \omega_1 + \omega_2$. Let $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_k k[[t]]$. Let $\pi e_1, \dots, \pi e_n$ denote the $k[[t]]$-basis of $\widetilde{\mathcal{E}}$ with the same pairing matrix. Set $\widetilde{\omega}_1$ generated by $\pi e_1, \dots, \pi e_a$, $\widetilde{\omega}_2$ by $\pi e_1, \pi e_\ell, \pi e_{a+1}, \dots, \pi e_{a+b-\ell}$ and $\widetilde{\omega}$ by $\pi e_1, \dots, \pi e_{n-1}, \pi e_n + te_1$, for a preimage by π of e_1 in $\widetilde{\mathcal{E}}$. We then check that $\widetilde{\omega}$ is totally isotropic : $< \pi e_n + te_1, \pi e_n + te_1 >=< \pi e_n, te_1 > + < te_1, \pi e_n >= 0$. Reducing all these data to $k[t]/(t^2)$ (which has divided powers over k) we deduce by Grothendieck-Messing and Serre-Tate a deformation of x to $k[t]/(t^2)$, and then inductively for all n to $k[t]/(t^n)$ (reducing this construction over $k[[t]]$), thus we get a $k[[t]]$-point of X, whose generic fiber has $h = 1$. □

Now, as the smooth locus of X is open, it cannot contain any of the $X_{h,\ell}$ with $\ell \neq 0$. Indeed, if $x \in X_{h,\ell}$ with $\ell \neq 0$ is in the smooth locus, any deformation of it also is in the smooth locus. But there is a deformation with $h \neq 0$ thus we can assume that $h \neq 0$ for x. Then any deformation of x has $h \neq 0$ and take a deformation y of x with maximal h and minimal $\ell \geq h$, thus $\ell \neq 0$, and y is still in the smooth locus. Up to change x by y, we can assume every deformation of x lies in $X_{h,\ell}$. Because the smooth locus is open, there is an open $U \subset X^{sm}$ containing Y. Reducing U if necessary, we can assume that U is irreducible and is included in $\cup_{h' \geq h, \ell' \leq \ell} X_{h',\ell'}$. But as there is no further possible deformation of x in X, actually U is included in $X_{h,\ell}$. But the smooth locus of $X_{h,\ell}$ is empty as $\ell \neq 0$[4]. □

[3]deform $\pi^{-1}(e_1 + e_2) \in \omega$ by $\pi^{-1}(e_1 + e_2) + t\pi^{-1}e_1$ and check it remains totally isotropic mod t^2

[4]Obviously if we prove the closure relations for the strata, then this argument simplifies a lot.

6.3 Geometry in the second case

In this section we assume that we are in the second case, i.e. the modified pairing given on $A[\pi]$ is given by the matrix

$$\begin{pmatrix} A & 0 & \dots & 0 \\ 0 & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & A \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Proposition 6.5. *The stratum $X_{h,l}$ is empty if l is not equal to a modulo 2.*

Proof. Let us consider the modified pairing on ω_1. It induces a non degenerate bilinear form on $\omega_1/\omega_1 \cap \omega_2$. The associated quadratic form on this space is identically zero; this implies that its dimension must be even. Since the dimension is equal to $a - l$, the result follows. □

Proposition 6.6. *The smooth locus consists in the strata $X_{h,h} \cup X_{h-1,h}$ for $1 \leq h \leq a$, with $h = a$ modulo 2, and $X_{0,0}$ if a is even. Each of the previous sets are open, of dimension $ab + h$ and ab respectively.*

Proof. It is clear that the points in $X_{0,0}$ are smooth. Let $1 \leq h \leq a$, with $a - h$ even. By the previous proposition, it is clear that $X_{h,h} \cup X_{h-1,h}$ is open. Let x be a point in $X_{h,h} \cup X_{h-1,h}$, and let us prove that it is a smooth point, by computing the tangent space. First let us remark that on $X_{h,h} \cup X_{h-1,h}$, the space $\omega_1 \cap \omega_2$ has rank h. Deforming x thus amounts to first deform the space $\omega_1 \cap \omega_2$ to a totally isotropic space $\mathcal{F}$; then deform ω_1 inside the orthogonal of $\mathcal{F}$. Finally, one should deform ω, contained in $\pi^{-1}\mathcal{F}$ and containing $\mathcal{F}^\perp$. Note that the original pairing descends to the quotient $\pi^{-1}\mathcal{F}/\mathcal{F}^\perp$. The second and third operations are smooth, of dimension respectively $(a-h)(b-h)$ and $h(h+1)/2$. We thus need to prove that the first operation is smooth of dimension $h(a+b-h) - h(h-1)/2$.
One can assume that the matrix of the modified pairing on $\mathcal{E}[\pi]$ is

$$\begin{pmatrix} 0 & 0 & I_h \\ 0 & B & 0 \\ I_h & 0 & 0 \end{pmatrix}$$

where this decomposition is written with respect to the inclusions $\omega_1 \cap \omega_2 \subseteq \omega_1 + \omega_2$, and B is a matrix with copies of the matrix A on the diagonal. Deforming the space $\omega_1 \cap \omega_2$ involves a matrix $\begin{pmatrix} I_h \\ X \\ Y \end{pmatrix}$, and thus $h(a + b - h)$ coordinates. The fact that this space should be totally isotropic gives the condition

$$Y +^t Y =^t XBX$$

This is an equality between symmetric matrices which have coefficients 0 on the diagonal. This is thus a smooth condition, with $h(h-1)/2$ linearly independent equations. We are thus left to prove that any point not in $\cup_{h \equiv a \pmod 2} X_{h,h} \cup X_{h-1,h}$ is not a smooth point. Let $x \in X_{h,\ell}(k)$ with $\ell - k \geq 2$. In particular $\dim \omega_1 \cap \omega_2/(\pi\omega) \geq 2$, thus let $\pi e_1, \pi e_2 \in \omega_1 \cap \omega_2 \subset \mathcal{E}[\pi]$ be two vectors, linearly independant when sent to $\omega_1 \cap \omega_2/(\pi\omega)$. As in the case of characteristics 2, we will look for specific lifts. As $\dim \omega[\pi] \backslash (\omega_1 + \omega_2) = a + b - h - (a + b - \ell) = \ell - h \geq 2$, we can find two linearly independant vectors $\pi e_{a+b-h-1}, \pi e_{a+b-h}$ there such that $\{\pi e_1, \pi e_{a+b-h-1}\} = 1 = \{ \pi e_2, \pi e_{a+b-h}\}$ and $\{\pi e_1, \pi e_{a+b-h}\} = 0 = \{ \pi e_2, \pi e_{a+b-h-1}\}$. Indeed, $\omega[\pi]$ is the orthogonal of $\pi\omega$ for the modified pairing $\{,\}$. Moreover modifiying $\pi e_{a+b-h-1}$ by $\pi e_{a+b-h-1} - t\pi e_{a+b-h}$ we can moreover assume $\{\pi e_{a+b-h-1}, \pi e_{a+b-h}\} = 0$ (the norm are automatically zero as we are in the second case). Assume that $\pi e_1, \dots, \pi e_n$ is a basis of $\mathcal{E}$ such that $\pi e_1, \dots, \pi e_a$ is a basis of ω_1, $\pi e_1, \dots, \pi e_\ell, \pi e_{a+1}, \dots, \pi e_{a+b-\ell}$ a basis of ω_2, and $\pi e_1, \dots, \pi e_{a+b-h}$ a basis of ω_π, and assume given lifts $e_\ell, \dots, e_{\ell-h+1}$ of $\pi e_\ell, \dots, \pi e_{\ell-h+1}$ inducing a basis of $\omega/\omega[\pi]$ (they are thus two by two orthogonal). We can also assume that $\{\pi e_{a+b-h-1}, \pi e_j\} = 0$ for all $j \neq 1$ and $\{\pi e_{a+b-h}, \pi e_j\} = 0$ for all $j \neq 2$ (up to modify by linear combination of the πe_j). Choose $e_1, e_2, e_{a+b-h-1}, e_{a+b-h} \in \mathcal{E}$ which are sent by π to $\pi e_1, \pi e_2, \pi e_{a+b-h-1}, \pi e_{a+b-h}$ and which are moreover two by two orthogonal together with $e_\ell, \dots, e_{\ell-h+1}$ (we can modify each by a π-torsion element). We set the following in $\widetilde{\mathcal{E}} = \mathcal{E} \otimes_k k[t]/(t^3)$:

$$\widetilde{\omega}_1 = (\pi e_1, \pi e_2 + t\pi e_{a+b-h-1}, \pi e_3, \dots, \pi e_a), \quad \text{and}$$

$$\widetilde{\omega} = (\pi e_1, \dots, \ \pi e_{a+b-h-2}, e_\ell, \dots, e_{\ell-h+1}, \pi e_{a+b-h-1} + te_1, \pi e_{a+b-h} + te_2 + t^2 e_{a+b-h-1}$$

Clearly, $\pi\widetilde{\omega} \subset \widetilde{\omega}_1$. We claim that modulo t^2 this defines a lift of x. We just need to check if ω is totally isotropic, and this boils down to

$$< \pi e_2 + t\pi e_{a+b-h-1}, \pi e_{a+b-h} + te_2 >= \{\pi e_2 + t\pi e_{a+b-h-1}, t\pi e_2\} = 0,$$

$$< \pi e_1, \pi e_{a+b-h-1} + te_1 >= \{\pi e_1, t\pi e_1\} = 0,$$

$$< \pi e_2 + t\pi e_{a+b-h-1}, \pi e_{a+b-h-1} + te_1 >= \{\pi e_2 + t\pi e_{a+b-h-1}, t\pi e_1\} = t^2 = 0,$$

$$< \pi e_1, \pi e_{a+b-h} + te_2 >= \{\pi e_1, t\pi e_2\} = 0,$$

Now assume we have another lift modulo t^3, this implies that there exists vectors $v_1 \in \widetilde{\omega_1}'$ with $v_1 = \pi e_1 + t^2 w_1$, $v_2 = \pi e_2 + t\pi e_{a+b-h-1} + t^2 w_2$, and $v_3, v_4 \in \widetilde{\omega}'$ such that $v_3 = \pi e_{a+b-h-1} + te_1 + t^2 w_3, v_4 = \pi e_{a+b-h} + te_2 + t^2 e_{a+b-h-1} + t^2 w_4$. Moreover, $t^2\pi w_3, t^2 \pi w_4$. But then, $\widetilde{\omega}'$ should be totally isotropic, but,

$$\begin{aligned} < v_2, v_3 > &=< \pi e_2 + t\pi e_{a+b-h-1} + t^2 w_2, \pi e_{a+b-h-1} + te_1 + t^2 w_3 > \\ &= \{\pi e_2 + t\pi e_{a+b-h-1} + t^2 v_2, t\pi e_1 + t^2 \pi w_3\} = t^2 \neq 0. \end{aligned}$$

□

Remark 6.7. If $a = 1$, the whole variety is smooth.

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